MAU23041: Advanced Classical Mechanics I
Homework 7 due 27/11/2020

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SF Theoretical Physics

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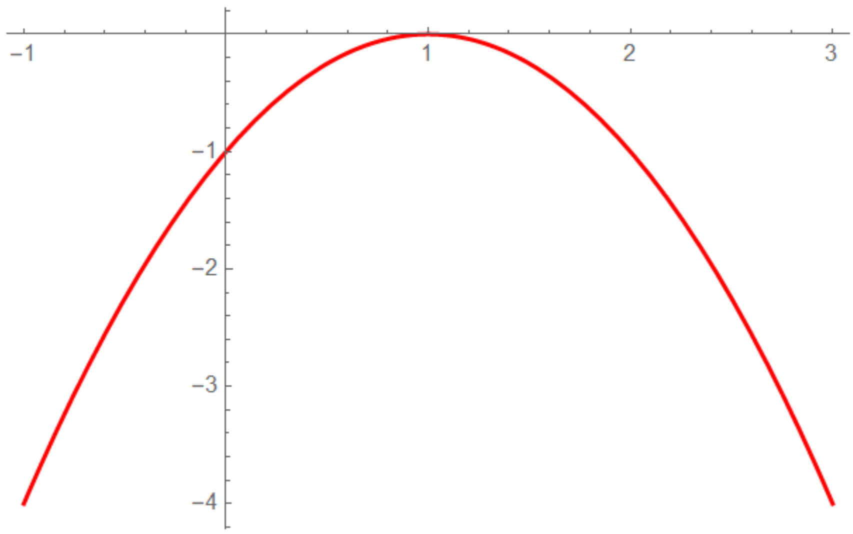
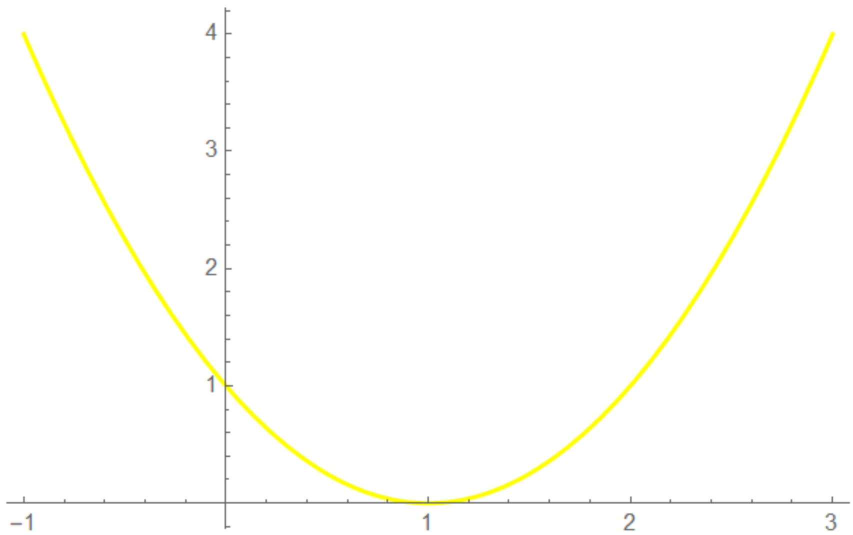
Problem 1

1.

```
a = 1;
Plot[-a x^2 + 2 a x - a, {x, -1, 3}, PlotStyle → {Thick, Red}]
Plot[a x^2 - 2 a x + a, {x, -1, 3}, PlotStyle → {Thick, Yellow}]
```





2.

$$
\begin{aligned}
L &= \frac{m}{2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) - \frac{k\Delta^2}{2}, \quad \Delta \equiv \text{spring length} \\
y &= -ax^2 + 2ax - a \\
&= -a(x-1)^2 \\
&= -aq^2, \quad q = x - 1 \\
\frac{dy}{dt} &= -2aq\dot{q} \\
\frac{dx}{dt} &= \dot{q} \\
\Delta &= \sqrt{q^2 + (y-l)^2} \\
&= \sqrt{q^2 + (-aq^2 - l)^2} \\
F &= kl \\
\implies k &= \frac{F}{l} \\
\implies L &= \frac{m}{2}\left(\dot{q}^2 + (-2aq\dot{q})^2\right) - \frac{F}{2l}\left(q^2 + (-aq^2 - l)^2\right) \\
L &= \frac{1}{2}\left(m\dot{q}^2\left(1 + 4a^2q^2\right) - \frac{F}{l}\left(q^2 + a^2q^4 + 2laq^2 + l^2\right)\right)
\end{aligned}
$$

3.

$$
\begin{aligned}
U &= \frac{k}{2}\left(q^2 + a^2q^4 + 2laq^2 + l^2\right) \\
\frac{dU}{dq} &= k\left(q + 2a^2q^3 + 2laq\right) \\
&= 0 \\
\implies q = 0,& \quad q = \pm\frac{1}{a}\sqrt{\frac{-2la-1}{2}} \\
\frac{d^2U}{dq^2} &= k\left(1 + 6a^2q^2 + 2la\right) \\
\left.\frac{d^2U}{dq^2}\right|_0 &= k(1 + 2la) \\
&> 0 \text{ if } a > -\frac{1}{2l} \\
\left.\frac{d^2U}{dq^2}\right|_{\pm\frac{1}{a}\sqrt{\frac{-2la-1}{2}}} &= k(-2 - 4la) \\
&> 0 \text{ if } a < -\frac{1}{2l}
\end{aligned}
$$

Equilibrium is stable if $\frac{dU}{dq} = 0$ and $\frac{d^2U}{dq^2} > 0$. Thus the stable equilibrium positions are $q = 0$ if $a > -\frac{1}{2l}$ and $q = \pm\frac{1}{a}\sqrt{\frac{-2la-1}{2}}$ if $a < -\frac{1}{2l}$.

4.

$$q_0 = 0 \qquad\qquad q_0 = \pm\frac{1}{a}\sqrt{\frac{-2la-1}{2}}$$

$$L = \frac{m\dot{q}^2}{2} - Fq^2\left(a+\frac{1}{2l}\right) - \frac{Fl}{2} \qquad\qquad L = -\frac{m\dot{q}^2(4al+1)}{2} + Fq^2\left(2a+\frac{1}{l}\right) + \frac{F(4al+1)}{8a^2l}$$

$$\omega = \sqrt{\frac{F+2aFl}{lm}} \qquad\qquad \omega = \sqrt{\frac{F+2aFl}{lm+4al^m}}$$

```
L = 1/2 (-F (q^2 + (l + a q^2)^2)/l + m (1 + 4 a^2 q^2) v^2);
L1 = Collect[(Series[(L /. {q → ϵ q, v → ϵ v}), {ϵ, 0, 2}] // Normal) /. ϵ → 1, {q, v}]
q0 = -Sqrt[-1 - 2 a l]/(Sqrt[2] a);
L2 = Collect[(Series[(L /. {q → ϵ q - q0, v → ϵ v}), {ϵ, 0, 2}] // Normal) /. ϵ → 1, {q, v}]
```

$$-\frac{F\,l}{2} + \left(-a\,F - \frac{F}{2\,l}\right) q^2 + \frac{m\,v^2}{2}$$

$$-\frac{F\,(-1-4\,a\,l)}{8\,a^2\,l} + \left(2\,a\,F + \frac{F}{l}\right) q^2 + \frac{1}{2}\,(-1-4\,a\,l)\,m\,v^2$$

```
ω1 = Sqrt[(a F + F/(2 l))/(m/2)] // FullSimplify
ω2 = Sqrt[(2 a F + F/l)/(m (4 a l + 1))] // FullSimplify
```

$$\sqrt{\frac{F + 2\,a\,F\,l}{l\,m}}$$

$$\sqrt{\frac{F + 2\,a\,F\,l}{l\,m + 4\,a\,l^2\,m}}$$

Problem 2

1.

$$
\begin{aligned}
ma + kx &= \frac{F_0}{\beta}\, e^{\gamma t} \\
\implies \ddot{x} + \omega^2 x - f\, e^{\gamma t} &= 0, \quad f \equiv \frac{F_0}{m\beta} \\
x(t) &= \zeta\, e^{\gamma t} \\
\implies \zeta &= \frac{f}{\gamma^2 + \omega^2} \\
t < 0 \implies x_-(t) &= \Im(x(t)), \quad \gamma = \alpha + i\beta \\
&= \frac{e^{\alpha t} F_0 \left(\left(\alpha^2 - \beta^2 + \omega^2\right)\sin(\beta t) - 2\alpha\beta\cos(\beta t)\right)}{m\beta\left(\alpha^2 + (\beta-\omega)^2\right)\left(\alpha^2 + (\beta+\omega)^2\right)} \\
x_-(0) &= \frac{F_0}{m\beta\left((\alpha+i\beta)^2 + \omega^2\right)} \\
\dot{x}_-(0) &= \frac{F_0(\alpha+i\beta)}{m\beta\left((\alpha+i\beta)^2 + \omega^2\right)} \\
\text{General solution: } x(t) &= A_+ e^{i\omega t} + A_- e^{-i\omega t} + \zeta\, e^{\gamma t} \\
t > 0 \implies x_+(t) &= x(t), \quad \zeta = \frac{f}{\gamma^2+\omega^2}, \quad \gamma = -\alpha + i\beta \\
&= A_+ e^{i\omega t} + A_- e^{-i\omega t} + \frac{F_0 e^{(-\alpha+i\beta)t}}{m\beta\left(\omega^2 + (-\alpha+i\beta)^2\right)} \\
x_+(0) &= A_+ + A_- + \frac{F_0}{m\beta\left(\omega^2 + (-\alpha+i\beta)^2\right)} \\
\dot{x}_+(0) &= i\omega A_+ - i\omega A_- + \frac{F_0(-\alpha+i\beta)}{m\beta\left(\omega^2 + (-\alpha+i\beta)^2\right)} \\
x_+(0) &= x_-(0) \\
v_+(0) &= v_-(0) \\
\implies A_+ &= -\frac{iF\alpha}{m\beta\omega\left(\alpha^2 + (\beta-\omega)^2\right)} \\
A_- &= \frac{iF\alpha}{m\beta\omega\left(\alpha^2 + (\beta+\omega)^2\right)} \\
x_+(t) &= \Im(x(t)) \\
&= \frac{F_0\left(e^{-\alpha t}\left(2\alpha\beta\cos(\beta t) + \left(\alpha^2 - \beta^2 + \omega^2\right)\sin(\beta t)\right) - 4\alpha\beta\cos(\omega t)\right)}{m\beta\left(\alpha^2 + (\beta-\omega)^2\right)\left(\alpha^2 + (\beta+\omega)^2\right)} \\
\implies x(t) &= \begin{cases} x_-, & t \le 0 \\ x_+, & t > 0 \end{cases} \\
&= \begin{cases} \frac{e^{\alpha t} F_0\left(\left(\alpha^2-\beta^2+\omega^2\right)\sin(\beta t) - 2\alpha\beta\cos(\beta t)\right)}{m\beta\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)}, & t \le 0 \\ \frac{F_0\left(e^{-\alpha t}\left(2\alpha\beta\cos(\beta t) + \left(\alpha^2-\beta^2+\omega^2\right)\sin(\beta t)\right) - 4\alpha\beta\cos(\omega t)\right)}{m\beta\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)}, & t > 0 \end{cases}
\end{aligned}
$$

```
EquationOfMotion = x''[t] + ω^2 x[t] - f e^(γ t)
```

$-e^{t\gamma} f + \omega^2 x[t] + x''[t]$

```
x[t_] = ξ e^(γ t);
Solve[EquationOfMotion == 0, ξ] // FullSimplify
```

$\left\{\left\{\xi \to \frac{f}{\gamma^2 + \omega^2}\right\}\right\}$

```
x1[t] = x[t] /. ξ → f/(γ^2 + ω^2) /. f → F/(m β) /. γ → α + i β
```

$$\frac{e^{t(\alpha + i\beta)} F}{m\beta\left((\alpha + i\beta)^2 + \omega^2\right)}$$

```
X1[t] = Assuming[{α > 0 && Im[α] == 0 && β > 0 && Im[β] == 0 && F > 0 && Im[F] == 0 && m > 0 && Im[m] == 0},
  (x1[t] - (x1[t] /. {i → -i}))/(2 i) // ExpToTrig // FullSimplify // Factor // FullSimplify]
```

$$\frac{e^{t\alpha} F\left(-2\alpha\beta \operatorname{Cos}[t\beta] + \left(\alpha^2 - \beta^2 + \omega^2\right)\operatorname{Sin}[t\beta]\right)}{m\beta\left(\alpha^2 + (\beta - \omega)^2\right)\left(\alpha^2 + (\beta + \omega)^2\right)}$$

```
x10 = F/(m β ((α + i β)^2 + ω^2))
v10 = D[x1[t], t] /. t → 0
```

$$\frac{F}{m\beta\left((\alpha + i\beta)^2 + \omega^2\right)}$$

$$\frac{F(\alpha + i\beta)}{m\beta\left((\alpha + i\beta)^2 + \omega^2\right)}$$

```
x[t_] = Aa e^(i ω t) + Ab e^(-i ω t) + ξ e^(γ t);
EquationOfMotion // FullSimplify
```

$$e^{t\gamma}\left(-f + \xi\left(\gamma^2 + \omega^2\right)\right)$$

```
xa[t_] = x[t] /. ξ → f/(γ^2 + ω^2) /. f → F/(m β) /. γ → -α + i β
```

$$\mathrm{Ab}\, e^{-i t\omega} + \mathrm{Aa}\, e^{i t\omega} + \frac{e^{t(-\alpha + i\beta)} F}{m\beta\left((-\alpha + i\beta)^2 + \omega^2\right)}$$

```
xa0 = xa[0]
va0 = D[xa[t], t] /. t → 0
```

$$Aa + Ab + \frac{F}{m\,\beta\,\left((-\alpha + i\,\beta)^2 + \omega^2\right)}$$

$$i\,Aa\,\omega - i\,Ab\,\omega + \frac{F\,(-\alpha + i\,\beta)}{m\,\beta\,\left((-\alpha + i\,\beta)^2 + \omega^2\right)}$$

```
Assuming[{α > 0 && Im[α] == 0 && β > 0 && Im[β] == 0 && F > 0 && Im[F] == 0 & m > 0 && Im[m] = 0},
  Solve[{x10 == xa0, v10 == va0}, {Aa, Ab}]] // FullSimplify
```

$$\left\{\left\{Aa \to -\frac{i\,F\,\alpha}{m\,\beta\,\left(\alpha^2 + (\beta - \omega)^2\right)\,\omega},\ Ab \to \frac{i\,F\,\alpha}{m\,\beta\,\omega\,\left(\alpha^2 + (\beta + \omega)^2\right)}\right\}\right\}$$

```
xda[t_] = xa[t] /. Aa → -(i F α)/(m β (α^2 + (β - ω)^2) ω) /. Ab → (i F α)/(m β ω (α^2 + (β + ω)^2))
```

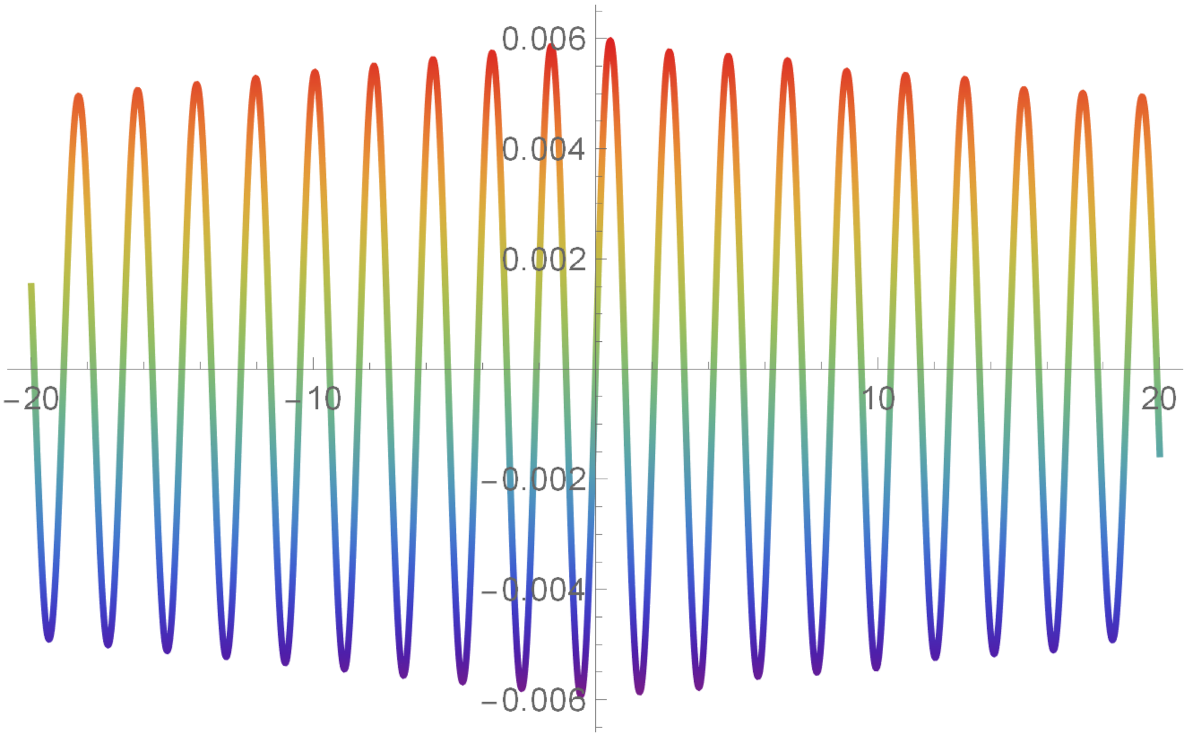
$$-\frac{i\,e^{i\,t\,\omega}\,F\,\alpha}{m\,\beta\,\left(\alpha^2 + (\beta - \omega)^2\right)\,\omega} + \frac{e^{t\,(-\alpha + i\,\beta)}\,F}{m\,\beta\,\left((-\alpha + i\,\beta)^2 + \omega^2\right)} + \frac{i\,e^{-i\,t\,\omega}\,F\,\alpha}{m\,\beta\,\omega\,\left(\alpha^2 + (\beta + \omega)^2\right)}$$

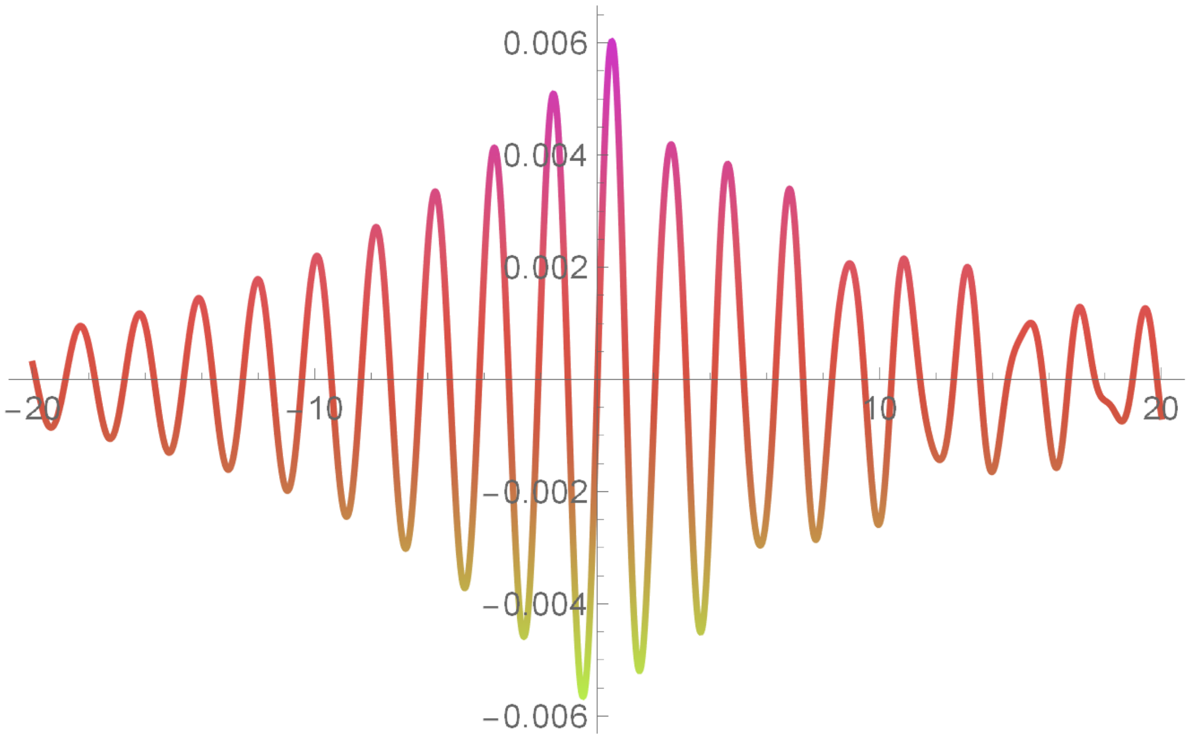
```
Xa[t_] =
 Assuming[{α > 0 && Im[α] == 0 && β > 0 && Im[β] == 0 && F > 0 && Im[F] == 0 & m > 0 && Im[m] = 0},
   Im[(xda[t] - (xda[t] /. {i → -i, -i → i}))/2]] // Expand // ExpToTrig // FullSimplify
```

$$Re\left[\frac{-4\,F\,\alpha\,\beta\,Cos[t\,\omega] + e^{-t\,\alpha}\,F\,\left(2\,\alpha\,\beta\,Cos[t\,\beta] + \left(\alpha^2 - \beta^2 + \omega^2\right)\,Sin[t\,\beta]\right)}{m\,\beta\,\left(\alpha^2 + (\beta - \omega)^2\right)\,\left(\alpha^2 + (\beta + \omega)^2\right)}\right]$$

2.

```
Xa[t_] = (-4 F α β Cos[t ω] + ⅇ^(-t α) F (2 α β Cos[t β] + (α^2 - β^2 + ω^2) Sin[t β])) / (m β (α^2 + (β - ω)^2) (α^2 + (β + ω)^2));
X1[t_] = (ⅇ^(t α) F (-2 α β Cos[t β] + (α^2 - β^2 + ω^2) Sin[t β])) / (m β (α^2 + (β - ω)^2) (α^2 + (β + ω)^2));
X[t_] = Piecewise[{{X1[t], t ≤ 0}, {Xa[t], t > 0}}];
Plot[X[t] /. {α → 0.01, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "Rainbow"]
Plot[X[t] /. {α → 0.1, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "NeonColors"]
Plot[X[t] /. {α → 0.25, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "CandyColors"]
Plot[X[t] /. {α → 0.5, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "FruitPunchColors"]
Plot[X[t] /. {α → 1, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "CMYKColors"]
Plot[X[t] /. {α → 10, F → 2, β → 3, ω → 5, m → 7}, {t, -20, 20}, PlotRange → All, PlotStyle → Thick,
 ColorFunction → "SandyTerrain"]
```

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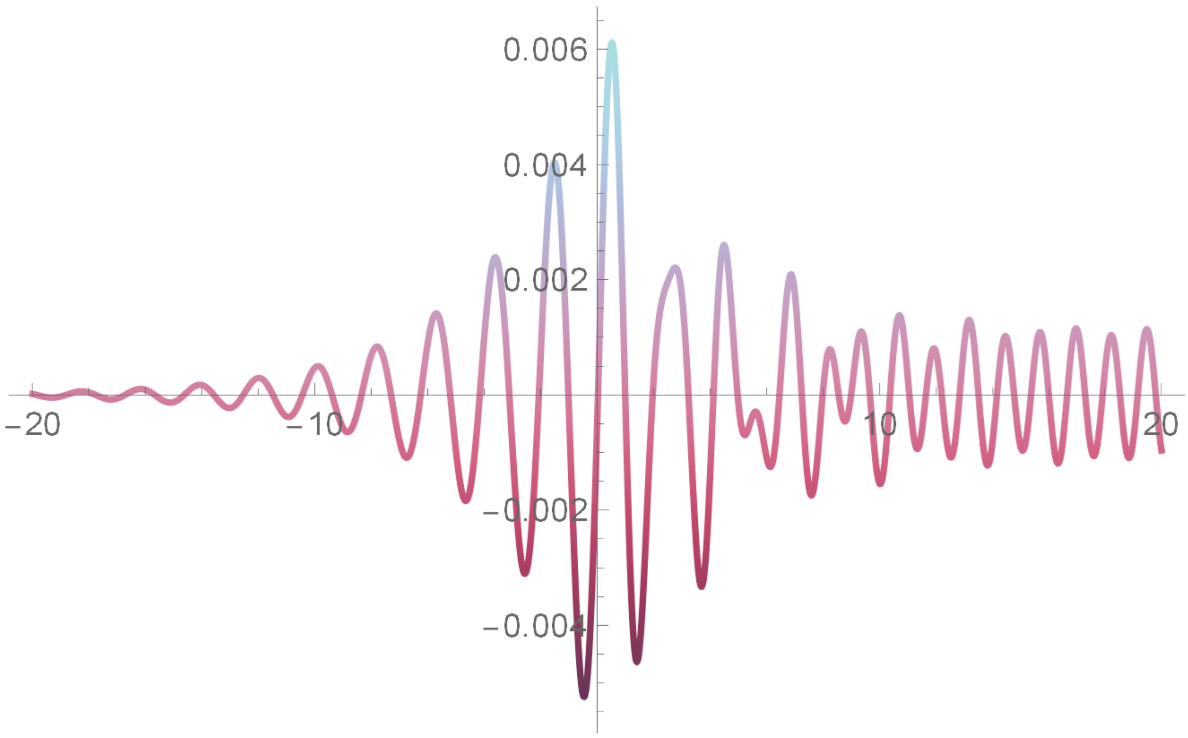
-10
10
20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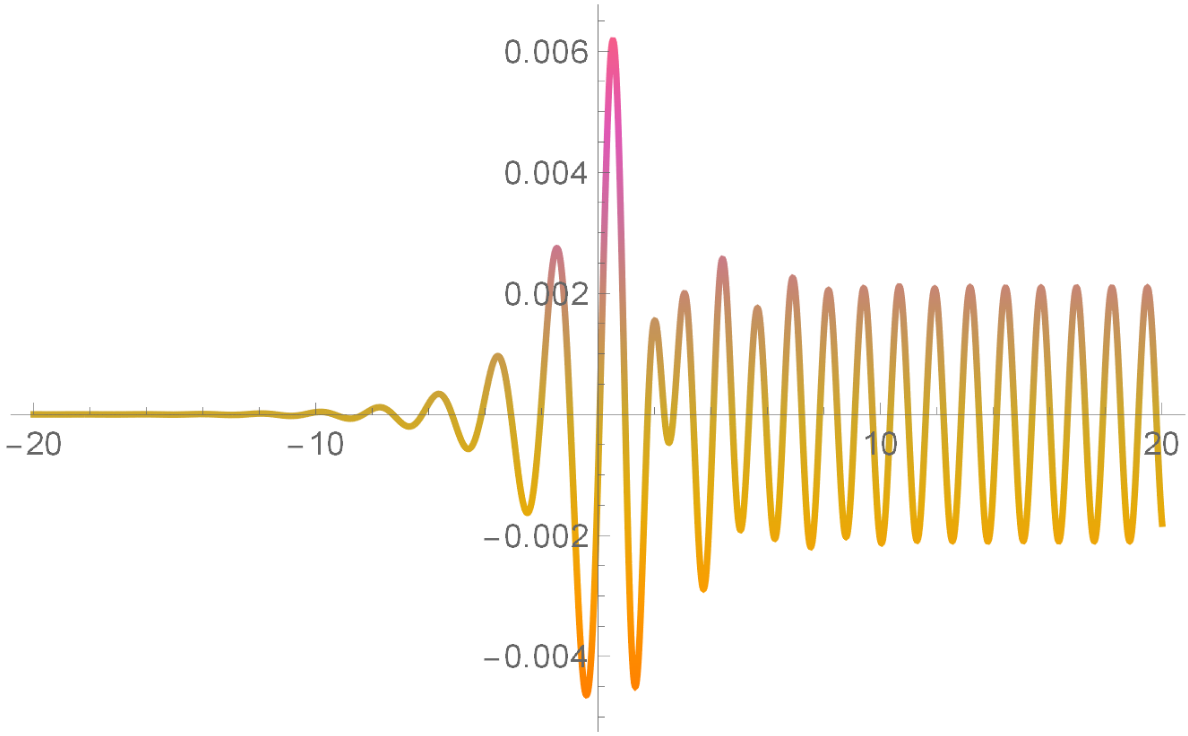
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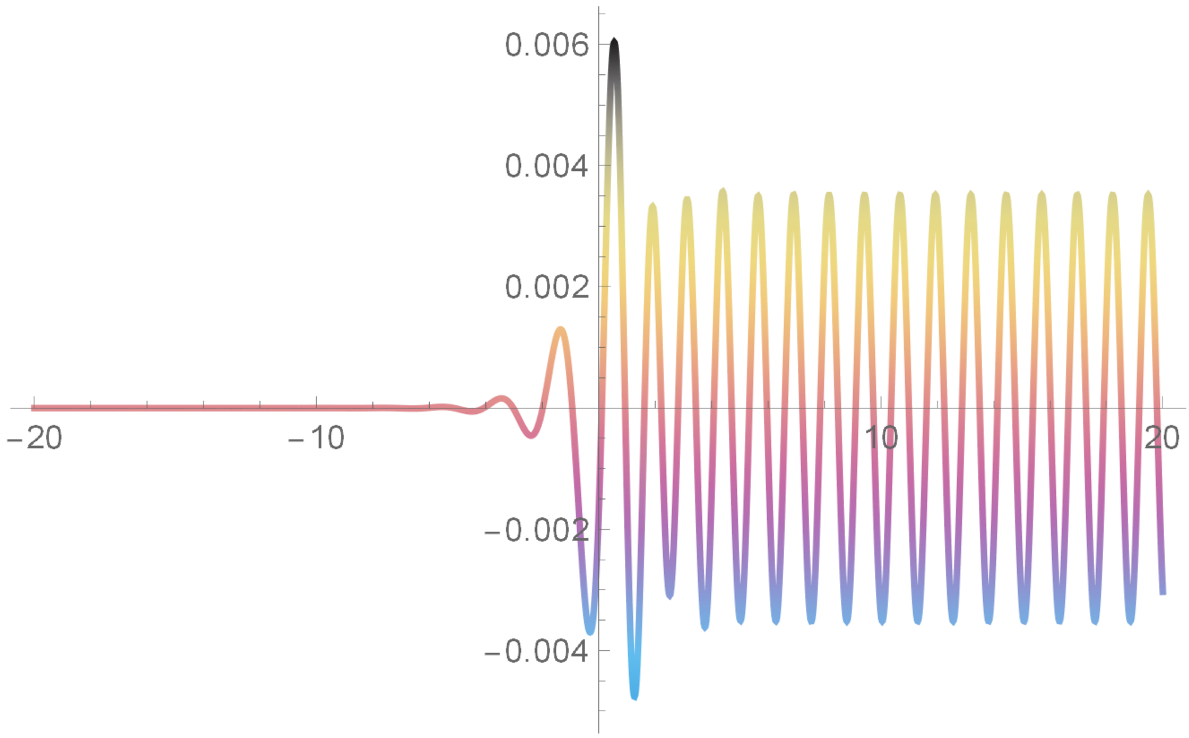
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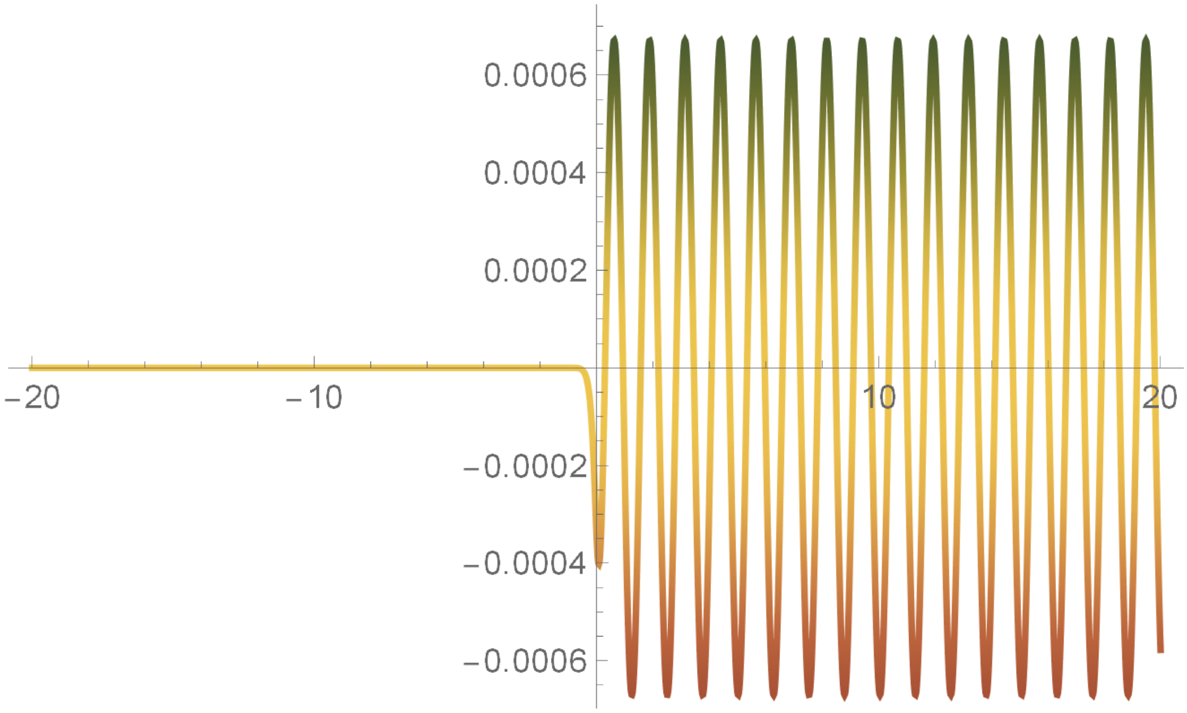

3.

Energy acquired by oscillator = Energy at infinity

$$x_\infty(t) = -\frac{4F_0\alpha\cos(\omega t)}{m\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)}$$

$$E_\infty = \frac{m\dot{x}_\infty{}^2}{2} + \frac{m\omega^2 x_\infty{}^2}{2}$$

$$E_\infty = \frac{8F_0{}^2\alpha^2\omega^2}{m\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)}$$

```
Xinfinity[t_] = (-4 F α β Cos[t ω]) / (m β (α^2 + (β - ω)^2) (α^2 + (β + ω)^2));
einfinity = (m (Xinfinity'[t])^2)/2 + (m ω^2 Xinfinity[t]^2)/2 // FullSimplify
```

$$\frac{8\,F^2\,\alpha^2\,\omega^2}{m\left(\alpha^2+(\beta-\omega)^2\right)^2\left(\alpha^2+(\beta+\omega)^2\right)^2}$$

4.

$$\lim_{\alpha\to 0} x_-(t) = -\frac{F_0 \sin(\beta t)}{m\beta\left(\beta^2-\omega^2\right)} \qquad \lim_{\beta\to 0} x_-(t) = \frac{F_0\, e^{\alpha t}\left(\omega^2 t + \alpha(\alpha t - 2)\right)}{m\left(\alpha^2+\omega^2\right)^2}$$

$$\lim_{\alpha\to 0} x_+(t) = -\frac{F_0 \sin(\beta t)}{m\beta\left(\beta^2-\omega^2\right)} \qquad \lim_{\beta\to 0} x_+(t) = \frac{F_0\, e^{-\alpha t}\left(\omega^2 t + \alpha(\alpha t + 2)\right) - 4F_0\alpha\cos(\omega t)}{m\left(\alpha^2+\omega^2\right)^2}$$

$$\lim_{\alpha\to 0} E_\infty = 0 \qquad \lim_{\beta\to 0} E_\infty = \frac{8{F_0}^2\alpha^2\omega^2}{m\left(\alpha^2+\omega^2\right)^4}$$

```
Clear[α, β, einfinity]
einfinity = 8 F^2 α^2 ω^2 / (m (α^2 + (β - ω)^2)^2 (α^2 + (β + ω)^2)^2);
Limit[X1[t], α → 0] // FullSimplify
Limit[Xa[t], α → 0] // FullSimplify
Limit[einfinity, α → 0] // FullSimplify
```

$$-\frac{F\,\mathrm{Sin}[t\,\beta]}{m\,\beta^3 - m\,\beta\,\omega^2}$$

$$-\frac{F\,\mathrm{Sin}[t\,\beta]}{m\,\beta^3 - m\,\beta\,\omega^2}$$

0

```
Clear[α, β, einfinity]
einfinity = 8 F^2 α^2 ω^2 / (m (α^2 + (β - ω)^2)^2 (α^2 + (β + ω)^2)^2);
Limit[X1[t], β → 0] // FullSimplify
Limit[Xa[t], β → 0] // FullSimplify
Limit[einfinity, β → 0] // FullSimplify
```

$$\frac{e^{t\,\alpha}\,F\left(\alpha\,(-2 + t\,\alpha) + t\,\omega^2\right)}{m\left(\alpha^2+\omega^2\right)^2}$$

$$\frac{e^{-t\,\alpha}\,F\left(\alpha\,(2 + t\,\alpha) + t\,\omega^2 - 4\,e^{t\,\alpha}\,\alpha\,\mathrm{Cos}[t\,\omega]\right)}{m\left(\alpha^2+\omega^2\right)^2}$$

$$\frac{8\,F^2\,\alpha^2\,\omega^2}{m\left(\alpha^2+\omega^2\right)^4}$$