MAU23401: Advanced Classical Mechanics I
Homework 6 due 20/11/2020

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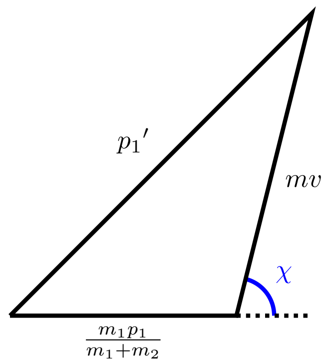
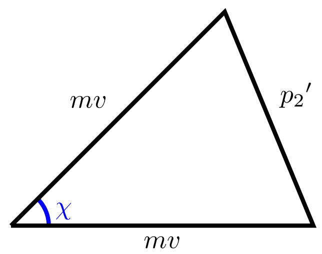
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

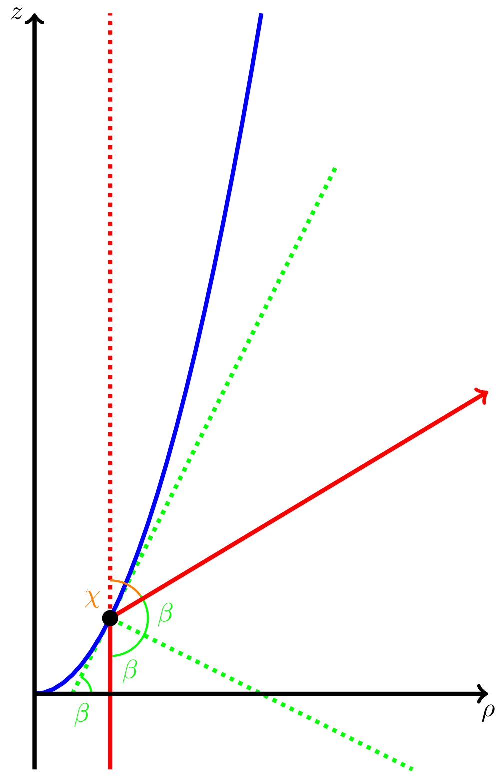


$$\begin{aligned}
{p_1'}^2 &= m^2v^2 + \frac{{m_1}^2{p_1}^2}{(m_1+m_2)^2} - \frac{2mvm_1p_1\cos(\pi-\chi)}{m_1+m_2} && \text{cosine rule}\\
&= \frac{{m_1}^2{m_2}^2v^2}{(m_1+m_2)^2} + \frac{{m_1}^4{v_1}^2}{(m_1+m_2)^2} + \frac{2{m_1}^2v_1vm_1m_2\cos\chi}{(m_1+m_2)^2} && \cos(\pi-x) = -\cos x\\
m_1{v_1'}^2 &= {m_1}^2\left(\frac{{m_2}^2v^2}{(m_1+m_2)^2} + \frac{{m_1}^2v^2}{(m_1+m_2)^2} + \frac{2m_1m_2v^2\cos\chi}{(m_1+m_2)^2}\right) && v = v_1 - v_2 = v_1\\
{v_1'}^2 &= \frac{{m_1}^2+{m_2}^2+2m_1m_2\cos\chi}{(m_1+m_2)^2}v^2\\
\implies v_1' &= \frac{\sqrt{{m_1}^2+{m_2}^2+2m_1m_2\cos\chi}}{m_1+m_2}v
\end{aligned}$$



$$\begin{aligned}
{p_2'}^2 &= m^2v^2 + m^2v^2 - 2m^2v^2\cos\chi && \text{cosine rule}\\
{m_2}^2{v_2'}^2 &= \frac{2{m_1}^2{m_2}^2v^2}{(m_1+m_2)^2}(1-\cos\chi)\\
{v_2'}^2 &= \frac{4{m_1}^2v^2}{(m_1+m_2)^2}\sin^2\left(\frac{\chi}{2}\right) && 2\sin^2 x = 1-\cos 2x\\
\implies v_2' &= \frac{2m_1v}{m_1+m_2}\sin\left(\frac{\chi}{2}\right)
\end{aligned}$$

Problem 2



$$
\begin{aligned}
z(x,y) &= a\left(x^2+y^2\right) \\
z\left(\rho\right) &= a\rho^2 \\
\text{Slope} &= \frac{dz}{d\rho} \\
&= 2a\rho \\
&= \tan\beta \\
\implies \rho\left(\beta\right) &= \frac{\tan\beta}{2a} \\
\chi &= \pi - 2\beta \\
\implies \rho\left(\chi\right) &= \frac{\tan\left(\frac{\pi}{2}-\frac{\chi}{2}\right)}{2a} \\
&= \frac{\cot\left(\frac{\chi}{2}\right)}{2a} \\
\frac{d\rho}{d\chi} &= \frac{-\csc^2\left(\frac{\chi}{2}\right)}{2a}\frac{1}{2} \\
\left|\frac{d\rho}{d\chi}\right| &= \frac{\csc^2\left(\frac{\chi}{2}\right)}{4a}
\end{aligned}
$$

$$
\begin{aligned}
d\sigma &= 2\pi\rho\left(\chi\right)\left|\frac{d\rho\left(\chi\right)}{d\chi}\right| d\chi \\
&= 2\pi\frac{\cot\left(\frac{\chi}{2}\right)}{2a}\frac{\csc^2\left(\frac{\chi}{2}\right)}{4a}d\chi \\
d\sigma &= \frac{\pi}{4a^2}\cot\left(\frac{\chi}{2}\right)\csc^2\left(\frac{\chi}{2}\right)d\chi
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
\rho^2 &= \frac{\alpha^2}{m^2 {v_\infty}^2} \cot^2\left(\frac{\chi}{2}\right) \\
2\rho \frac{d\rho}{d\chi} &= \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \left(2\cot\left(\frac{\chi}{2}\right)\right) \left(-\frac{1}{\sin^2\left(\frac{\chi}{2}\right)}\right) \left(\frac{1}{2}\right) \\
2\pi\rho \frac{d\rho}{d\chi} d\chi &= -\pi \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \left(\frac{\cos\left(\frac{\chi}{2}\right)}{\sin^3\left(\frac{\chi}{2}\right)}\right) d\chi \\
\left|2\pi\rho \frac{d\rho}{d\chi} d\chi\right| &= \left|-\pi \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \left(\frac{\cos\left(\frac{\chi}{2}\right)}{\sin^3\left(\frac{\chi}{2}\right)}\right) d\chi\right| \\
2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi &= \pi \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \left(\frac{\cos\left(\frac{\chi}{2}\right)}{\sin^3\left(\frac{\chi}{2}\right)}\right) d\chi \\
d\sigma &= \pi \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \frac{\cos\left(\frac{\chi}{2}\right)}{\sin^3\left(\frac{\chi}{2}\right)} d\chi \qquad\qquad d\sigma = 2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi
\end{aligned}
$$

$$
\begin{aligned}
{v_2}' &= \frac{2 m_1 v_\infty}{m_1 + m_2} \sin\left(\frac{\chi}{2}\right) \qquad\qquad \text{from Problem 1} \\
&= \frac{2 m v_\infty}{m_2} \sin\left(\frac{\chi}{2}\right) \\
\epsilon &\equiv \text{energy lost by } m_1 \\
&= \text{energy gained by } m_2 \\
&= \frac{1}{2} m_2 {v_2}'^2 \sin^2\left(\frac{\chi}{2}\right) \\
&= \frac{2 m^2 {v_\infty}^2}{m_2} \sin^2\left(\frac{\chi}{2}\right) \\
\frac{d\epsilon}{d\chi} &= \frac{2 m^2 {v_\infty}^2}{m_2} \left(2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\left(\frac{1}{2}\right)\right) \\
\implies d\chi &= \frac{m_2}{2 m^2 {v_\infty}^2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)} d\epsilon
\end{aligned}
$$

$$
\begin{aligned}
\implies d\sigma &= \pi \left(\frac{\alpha}{m {v_\infty}^2}\right)^2 \frac{\cos\left(\frac{\chi}{2}\right)}{\sin^3\left(\frac{\chi}{2}\right)} \frac{m_2}{2 m^2 {v_\infty}^2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)} d\epsilon \\
&= \frac{\pi m_2}{2\sin^4\left(\frac{\chi}{2}\right)} \left(\frac{\alpha}{m^2 {v_\infty}^3}\right)^2 d\epsilon
\end{aligned}
$$

$$
\begin{aligned}
\epsilon &= \frac{2 m^2 {v_\infty}^2}{m_2} \sin^2\left(\frac{\chi}{2}\right) \\
\implies \sin^2\left(\frac{\chi}{2}\right) &= \frac{\epsilon m_2}{2 m^2 {v_\infty}^2}
\end{aligned}
$$

$$
\begin{aligned}
\implies d\sigma &= \frac{\pi m_2}{2} \left(\frac{\alpha}{m^2 {v_\infty}^3}\right)^2 \left(\frac{2 m^2 {v_\infty}^2}{\epsilon m_2}\right)^2 d\epsilon \\
&= \frac{2\pi}{m_2} \left(\frac{\alpha^2}{v_\infty}\right)^2 \frac{d\epsilon}{\epsilon}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
U &= \kappa\left(r^2+a^2\right)^{-\frac{1}{2}} \\
\frac{dU}{dr} &= -\frac{\kappa}{2}\left(r^2+a^2\right)^{-\frac{3}{2}}(2r) \\
&= -\frac{\kappa r}{\left(r^2+a^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= -\frac{2\rho}{m_1 {v_\infty}^2}\int_\rho^\infty \frac{dU}{dr}\frac{dr}{\sqrt{r^2-\rho^2}} \\
\theta &= \frac{2\kappa\rho}{m{v_\infty}^2}\int_\rho^\infty \frac{r\,dr}{\left(r^2+a^2\right)^{\frac{3}{2}}\sqrt{r^2-\rho^2}} \\
&= \frac{2\kappa\rho}{m{v_\infty}^2\left(a^2+\rho^2\right)} \\
\rho &= \frac{\kappa \pm \sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}{m\theta{v_\infty}^2} \\
\rho &\neq \frac{\kappa - \sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}{m\theta{v_\infty}^2} \qquad \text{as if } a=0,\ \rho=0 \\
\implies \rho &= \frac{\kappa + \sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}{m\theta{v_\infty}^2} \\
\frac{d\rho}{d\theta} &= -\frac{a^2m{v_\infty}^2}{\sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}} - \frac{\kappa + \sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}{m\theta^2{v_\infty}^2} \\
\left|\frac{d\rho}{d\theta}\right| &= \frac{a^2m{v_\infty}^2}{\sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}} + \frac{\kappa + \sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}{m\theta^2{v_\infty}^2} \\
d\sigma &= \left|\frac{d\rho}{d\theta}\right|\frac{\rho(\theta)}{\theta}\mathrm{do} \\
&= 2\pi\rho\left|\frac{d\rho}{d\theta}\right|d\theta \qquad \text{as do} = 2\pi\sin\theta\,d\theta \approx 2\pi\theta\,d\theta \\
d\sigma &= \frac{2\pi\kappa}{m^2\theta^3{v_\infty}^4}\frac{\left(\kappa+\sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}\right)^2}{\sqrt{\kappa^2-a^2m^2\theta^2{v_\infty}^4}}
\end{aligned}
$$

$$\frac{2\kappa\rho}{\mathrm{m}\, \mathrm{v}_\infty{}^2}\ \mathbf{Assuming}\left[\{\mathrm{Im}[\kappa] == 0\ \&\&\ \mathrm{Im}[\rho] == 0\ \&\&\ \mathrm{Im}[\mathrm{m}] == 0\ \&\&\ \mathrm{Im}[\mathrm{v}_\infty] == 0\ \&\&\ \mathrm{Im}[\mathrm{a}] == 0\},\ \mathbf{Integrate}\left[\frac{r}{(r^2+a^2)^{\frac{3}{2}}\,(r^2-\rho^2)^{\frac{1}{2}}},\ \{r, \rho, \infty\}\right]\right]$$

$$\boxed{\frac{2\kappa\rho}{\mathrm{m}\,(a^2+\rho^2)\,\mathrm{v}_\infty^2}\quad \text{if}\ \ a \neq 0\ \&\&\ \rho > 0}$$

$$\mathbf{Solve}\left[\frac{2\kappa\rho}{\mathrm{m}\,\mathrm{v}_\infty{}^2\,(a^2+\rho^2)} == \theta,\ \rho\right]$$

$$\left\{\left\{\rho \to \frac{\kappa - \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta\, v_\infty^2}\right\},\ \left\{\rho \to \frac{\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta\, v_\infty^2}\right\}\right\}$$

$$\mathrm{D}\left[\frac{\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta\, v_\infty^2},\ \theta\right]$$

$$-\frac{a^2\, m\, v_\infty^2}{\sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}} - \frac{\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta^2\, v_\infty^2}$$

$$\rho = \frac{\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta\, v_\infty^2};$$

$$\mathrm{d}\rho\mathrm{d}\theta = -\frac{a^2\, m\, v_\infty^2}{\sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}} - \frac{\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}{m\,\theta^2\, v_\infty^2};$$

$$\mathrm{d}\sigma = \mathbf{FullSimplify}[-2\,\pi\,\rho\ \mathrm{d}\rho\mathrm{d}\theta\ \mathrm{d}\theta]$$

$$\frac{2\,\mathrm{d}\theta\,\pi\,\kappa\left(\kappa + \sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}\right)^2}{m^2\,\theta^3\, v_\infty^4\,\sqrt{\kappa^2 - a^2 m^2 \theta^2 v_\infty^4}}$$