MAU23401: Advanced Classical Mechanics I
Homework 5 due 06/11/2020

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SF Theoretical Physics

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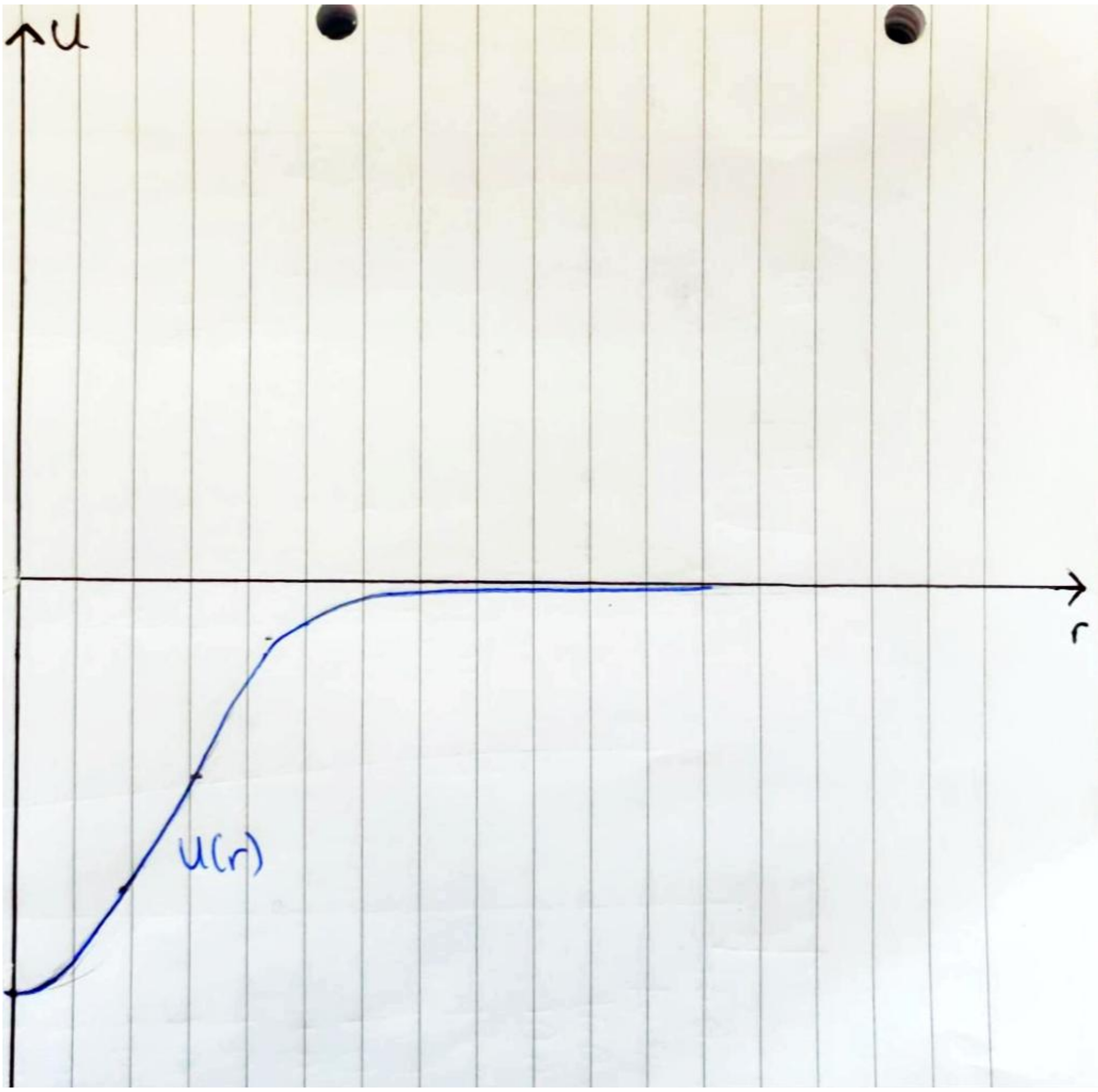
Problem 1

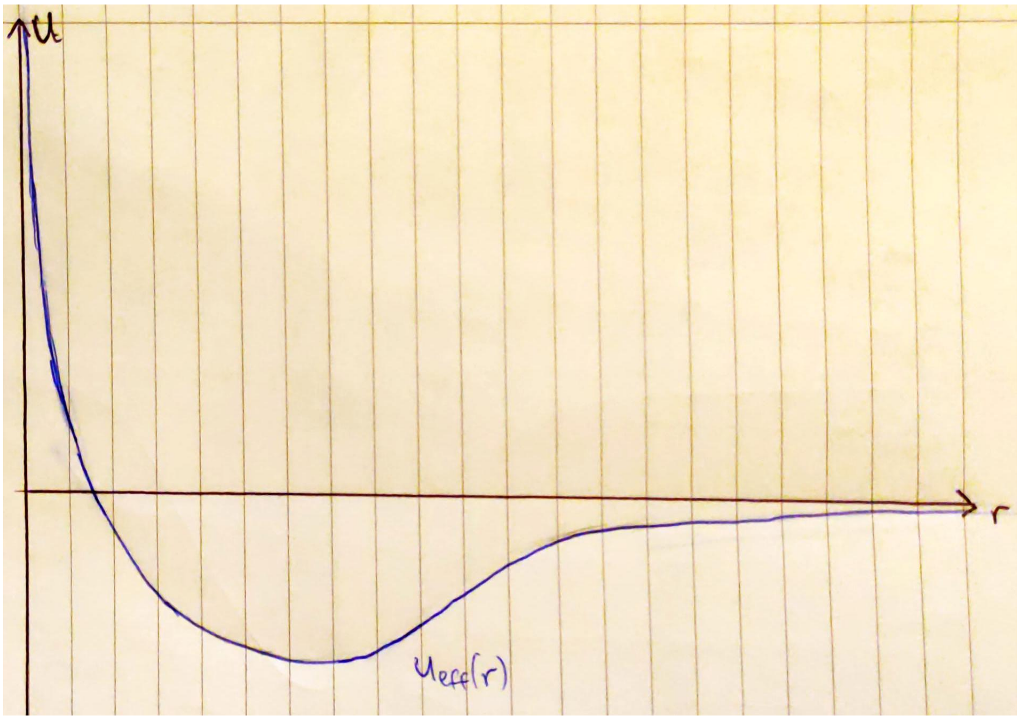
(a)



The field is attractive everywhere, as $F = -\dfrac{dU}{dr} < 0 \; \forall \; r > 0$.

(b)

$$U_{\text{eff}}(r) = U(r) + \frac{M^2}{2mr^2}$$
$$= \frac{M^2}{2mr^2} - \frac{\alpha^2}{2m\left(\beta^2 + r^2\right)}$$



Finite motion is possible. The particle is bounded by the annulus of radii $r_{\min}$ and $r_{\max}$. It may not necessarily be a closed loop.

Infinite motion is also possible. The particle can travel from infinity and get deflected around the origin, and continue to infinity.

$$U_{\text{eff}}(r_{\text{turning}}) = E$$

$$\text{Finite motion} \implies r_{\min} = \frac{1}{2}\sqrt{\frac{M^2 - \alpha^2}{Em} - 2\beta^2 + \frac{\sqrt{8EmM^2\beta^2 + \left(\alpha^2 + 2Em\beta^2 - M^2\right)^2}}{Em}}$$

$$\text{and } r_{\max} = \frac{1}{2}\sqrt{\frac{M^2 - \alpha^2}{Em} - 2\beta^2 - \frac{\sqrt{8EmM^2\beta^2 + \left(\alpha^2 + 2Em\beta^2 - M^2\right)^2}}{Em}}$$

$$\text{Infinite motion} \implies r_{\min} = \frac{1}{2}\sqrt{\frac{M^2 - \alpha^2}{Em} - 2\beta^2 - \frac{\sqrt{8EmM^2\beta^2 + \left(\alpha^2 + 2Em\beta^2 - M^2\right)^2}}{Em}}$$

```
Assuming[{r ≥ 0 && m > 0 && α > 0 && β > 0 && e < 0}, Solve[M^2/(2 m * r^2) - α^2/(2 m * (β^2 + r^2)) == e, r]]
```

$$\left\{\left\{r \to -\frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2-\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\left\{r \to \frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2-\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\right.$$

$$\left.\left\{r \to -\frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2+\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\left\{r \to \frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2+\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\}\right\}$$

```
Assuming[{r ≥ 0 && m > 0 && α > 0 && β > 0 && e > 0}, Solve[M^2/(2 m * r^2) - α^2/(2 m * (β^2 + r^2)) == e, r]]
```

$$\left\{\left\{r \to -\frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2-\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\left\{r \to \frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2-\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\right.$$

$$\left.\left\{r \to -\frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2+\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\},\left\{r \to \frac{1}{2}\sqrt{\frac{M^2}{e\,m}-\frac{\alpha^2}{e\,m}-2\beta^2+\frac{\sqrt{8\,e\,m\,M^2\beta^2+\left(-M^2+\alpha^2+2\,e\,m\,\beta^2\right)^2}}{e\,m}}\right\}\right\}$$

(c)

$$\left.\frac{dU_{\text{eff}}(r)}{dr}\right|_{r=r_0} = 0$$

$$\implies r_0 = \beta\sqrt{\frac{M}{\alpha - M}}$$

$$E = U_{\text{eff}}(r_0)$$

$$= \frac{M(-M+\alpha)}{2m\beta^2} - \frac{\alpha^2}{2m\left(\beta^2 + \frac{M\beta^2}{-M+\alpha}\right)}$$

$$E = -\frac{(\alpha - M)^2}{2m\beta^2}$$

$$\text{Ellipse} \implies 2m\pi ab = TM$$

$$\text{Circle} \implies 2m\pi {r_0}^2 = TM$$

$$T = \frac{2m\pi {r_0}^2}{M}$$

$$T = \frac{2m\pi\beta^2}{\alpha - M}$$

```
Ueff[r_] := M^2/(2 m * r^2) - α^2/(2 m * (β^2 + r^2));
Solve[D[Ueff[r], r] == 0, r]
```

$$\left\{\left\{r \to -\frac{\sqrt{M}\,\beta}{\sqrt{-M-\alpha}}\right\}, \left\{r \to \frac{\sqrt{M}\,\beta}{\sqrt{-M-\alpha}}\right\}, \left\{r \to -\frac{\sqrt{M}\,\beta}{\sqrt{-M+\alpha}}\right\}, \left\{r \to \frac{\sqrt{M}\,\beta}{\sqrt{-M+\alpha}}\right\}\right\}$$

```
r0 = (Sqrt[M] β)/Sqrt[-M + α];
Ueff[r0]
```

$$\frac{M\,(-M+\alpha)}{2\,m\,\beta^2} - \frac{\alpha^2}{2\,m\left(\beta^2 + \frac{M\,\beta^2}{-M+\alpha}\right)}$$

(d)

$$E < 0$$

$$\phi(r) = \text{constant}$$

$$M = 0 \text{ as } \vec{r} \parallel \vec{v} \text{ and } \vec{M} = \vec{r} \times m\vec{v}$$

$$\implies U_{\text{eff}}(r) = U(r)$$

$$E = U(A),\; A \equiv \text{amplitude}$$

$$E = -\frac{\alpha^2}{2m\left(\beta^2 + A^2\right)}$$

$$A = \sqrt{\frac{-\alpha^2 - 2Em\beta^2}{2Em}}$$

$$T = \sqrt{2m}\int_{x_a}^{x_b} \frac{dx}{\sqrt{E - U(x)}}$$

$$= 2\sqrt{2m}\int_A^0 \frac{dr}{\sqrt{\frac{\alpha^2}{2m(\beta^2+r^2)} - \frac{\alpha^2}{2m(\beta^2+A^2)}}}$$

$$T = \frac{4m\beta\sqrt{\beta^2 + A^2}}{\alpha}\text{EllipticE}\left(-\frac{A^2}{\beta^2}\right)$$

$$\approx \frac{2m\pi\beta^2}{\alpha} \text{ for small } A$$

For small amplitudes, the period does not depend on the amplitude. This makes sense, as the motion will be approximately simple harmonic, for which the period does not depend on amplitude.

```
Clear[r, e]
Solve[e == -α^2/(2 m * (β^2 + r^2)), r]
```

$$\left\{\left\{r \to -\frac{\sqrt{-\alpha^2 - 2\,e\,m\,\beta^2}}{\sqrt{2}\,\sqrt{e}\,\sqrt{m}}\right\}, \left\{r \to \frac{\sqrt{-\alpha^2 - 2\,e\,m\,\beta^2}}{\sqrt{2}\,\sqrt{e}\,\sqrt{m}}\right\}\right\}$$

```
Clear[r, e];
A = Sqrt[-α^2 - 2 e m β^2]/(Sqrt[2] Sqrt[e] Sqrt[m]);
2 Sqrt[2 m] Assuming[{α > 0 && β > 0 && m > 0 && e < 0 && -α^2 - 2 e * m * β^2 > 0}, Integrate[1/Sqrt[α^2/(2 m * (β^2 + r^2)) - α^2/(2 m * (β^2 + A^2))], {r, A, 0}]]
```

$$\frac{2\sqrt{2}\,\sqrt{m}\,\beta\,\text{EllipticE}\left[1 + \frac{\alpha^2}{2\,e\,m\,\beta^2}\right]}{\sqrt{-e}} \quad \text{if } 2\,e\,m + \alpha^2 > 0$$

```
Clear[e, A];
e = -α^2/(2 m * (β^2 + A^2));
2 Sqrt[2] Sqrt[m] β EllipticE[1 + α^2/(2 e m β^2)]/Sqrt[-e] // FullSimplify
```

$$\frac{4\sqrt{m}\,\beta\,\text{EllipticE}\left[-\frac{A^2}{\beta^2}\right]}{\sqrt{\frac{\alpha^2}{m\,(A^2+\beta^2)}}}$$

```
Clear[A];
Series[4 Sqrt[m] β EllipticE[-A^2/β^2]/Sqrt[α^2/(m (A^2 + β^2))], {A, 0, 0}] // Normal // FullSimplify
```

$$\frac{2\sqrt{m}\,\pi\,\beta}{\sqrt{\frac{\alpha^2}{m\,\beta^2}}}$$

Problem 2

(a)

$$\begin{aligned}
\mathcal{L} &= \frac{m}{2}v^2 - U(r) && \text{Lagrangian for the particle} \\
&= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) && \text{in polar coordinates} \\
M &= mr^2\dot{\phi} = \text{constant} && \phi \text{ is cyclic} \\
\implies \dot{\phi} &= \frac{M}{mr^2} \\
E &= T + U \\
&= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \\
&= \frac{m}{2}\dot{r}^2 + \frac{M^2}{2mr^2} + U(r) \\
&= \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + U_{\text{eff}}(r) && U_{\text{eff}}(r) \equiv U(r) + \frac{M^2}{2mr^2} \\
\frac{dr}{dt} &= \pm\sqrt{\frac{2(E - U_{\text{eff}}(r))}{m}} \\
&= \text{sign}(t)\sqrt{\frac{2(E - U_{\text{eff}}(r))}{m}} && \text{positive } t \implies +, \text{ negative } t \implies - \\
\frac{d\phi}{dr} &= \frac{d\phi}{dt}\frac{dt}{dr} \\
&= \dot{\phi}\left(\frac{dr}{dt}\right)^{-1} \\
&= \frac{M}{mr^2}\text{sign}(t)\sqrt{\frac{m}{2(E - U_{\text{eff}}(r))}} \\
&= \frac{M}{\sqrt{2m}}\text{sign}(t)\frac{1}{r^2\sqrt{E - U_{\text{eff}}(r)}} \\
\phi &= \frac{M}{\sqrt{2m}}\int_{r_a}^{r_b}\frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}} \\
\phi_0 &= \frac{M}{\sqrt{2m}}\int_{r_{\min}}^{\infty}\frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}} \\
\chi &\equiv |\pi - 2\phi_0| \\
&= \left|\pi - M\sqrt{\frac{2}{m}}\int_{r_{\min}}^{\infty}\frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}}\right|
\end{aligned}$$

(b)

$$\begin{aligned}
E &= \frac{m}{2}{v_\infty}^2 + U_\infty && \text{where } v_\infty \equiv \text{velocity at infinity} \\
M &= m\rho v_\infty && \text{and } U_\infty \equiv U(r = \infty) \\
\implies \chi &= \left|\pi - m\rho v_\infty\sqrt{\frac{2}{m}}\int_{r_{\min}}^{\infty}\frac{dr}{r^2\sqrt{\frac{m}{2}{v_\infty}^2 + U_\infty - U_{\text{eff}}(r)}}\right| \\
&= \left|\pi - 2\rho\int_{r_{\min}}^{\infty}\frac{dr}{r^2\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2}{m{v_\infty}^2}\left(U(r) - U_\infty\right)}}\right|
\end{aligned}$$

(c)

(i)

$$\begin{aligned}
U_{\text{eff}}(r) &= U(r) + \frac{M^2}{2mr^2} \\
&= \frac{\alpha + \beta r^2}{r^2} + \frac{M^2}{2mr^2} \\
U_{\text{eff}}(r_{\min}) &= E \\
\implies r_{\min} &= \sqrt{\frac{M^2 + 2m\alpha}{2Em - 2m\beta}} \\
\phi(r) &= \frac{M}{\sqrt{2m}} \int_{r_{\min}}^{r} \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}} \\
&= \frac{M}{\sqrt{M^2 + 2m\alpha}} \left(\frac{\pi}{2} + \operatorname{arccsc}\left(r\sqrt{\frac{2m(E-\beta)}{M^2 + 2m\alpha}} \right) \right) \\
\implies r(\phi) &= \sqrt{\frac{M^2 + 2m\alpha}{2m(E-\beta)}} \sec\left(\phi\, \frac{\sqrt{M^2 + 2m\alpha}}{M} \right) \\
t(r) &= \pm\sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \\
&= \sqrt{\frac{m}{2}} \int_{r_{\min}}^{r} \frac{dr}{\sqrt{E - \left(\frac{\alpha + \beta r^2}{r^2} + \frac{M^2}{2mr^2} \right)}} \\
&= \frac{r}{E - \beta} \sqrt{\frac{Em}{2} - \frac{M^2 + 2m\left(\alpha + r^2\beta\right)}{4r^2}} \\
\implies r(t) &= \sqrt{\frac{M^2 + 2m\alpha + 4t^2(E-\beta)^2}{2m(E-\beta)}} \\
\text{and } \phi(t) &= \frac{M}{\sqrt{M^2 + 2m\alpha}} \left(\pi - \operatorname{arcsec}\left(\sqrt{\frac{4t^2(E-\beta)^2}{M^2 + 2m\alpha} + 1} \right) \right)
\end{aligned}$$

$$\begin{aligned}
M &= m\rho v_\infty \\
E &= \frac{m}{2} {v_\infty}^2 + U_\infty \\
&= \frac{m}{2} {v_\infty}^2 + U(r = \infty) \\
&= \frac{m}{2} {v_\infty}^2 + \beta \\
\implies r(\phi) &= \rho\sqrt{1 + \frac{2\alpha}{m\rho^2 {v_\infty}^2}} \sec\left(\phi\sqrt{1 + \frac{2\alpha}{m\rho^2 {v_\infty}^2}} \right) \\
\text{and } r(t) &= \rho\sqrt{1 + \frac{2\alpha}{m\rho^2 {v_\infty}^2} + \frac{{v_\infty}^2 t^2}{\rho^2}} \\
\text{and } \phi(t) &= \frac{1}{\sqrt{1 + \frac{2\alpha}{m\rho^2 {v_\infty}^2}}} \left(\pi - \operatorname{arcsec}\left(\frac{1}{\rho}\sqrt{\frac{{v_\infty}^2 t^2}{1 + \frac{2\alpha}{m\rho^2 {v_\infty}^2}} + \rho^2} \right) \right)
\end{aligned}$$

$$\text{Assuming}\left[\{\alpha>0\,\&\&\,\beta>0\,\&\&\,m>0\},\ \text{Solve}\left[\frac{\alpha+\beta*r^2}{r^2}+\frac{M^2}{2\,m*r^2}==e,\ r\right]\right]$$

$$\left\{\left\{r\to-\frac{\sqrt{-M^2-2\,m\,\alpha}}{\sqrt{-2\,e\,m+2\,m\,\beta}}\right\},\ \left\{r\to\frac{\sqrt{-M^2-2\,m\,\alpha}}{\sqrt{-2\,e\,m+2\,m\,\beta}}\right\}\right\}$$

$$\text{Clear}[r,\ e]$$

$$\frac{M}{\sqrt{2\,m}}\ \text{Assuming}\left[\{\alpha>0\,\&\&\,\beta>0\,\&\&\,m>0\},\ \text{Integrate}\left[\frac{1}{q^2\sqrt{e-\left(\frac{\alpha+\beta*q^2}{q^2}+\frac{M^2}{2\,m*q^2}\right)}},\ \left\{q,\ \frac{\sqrt{-M^2-2\,m*\alpha}}{\sqrt{-2\,e*m+2\,m*\beta}},\ r\right\}\right]\right]$$

$$\frac{M\left(\pi+2\,\text{ArcCsc}\left[\sqrt{2}\,r\,\sqrt{\frac{m\,(e-\beta)}{M^2+2\,m\,\alpha}}\right]\right)}{\sqrt{2}\,\sqrt{m}\,\sqrt{\frac{2\,M^2}{m}+4\,\alpha}}\quad\text{if}\quad \textit{condition}$$

$$\text{Assuming}\left[\{\alpha>0\,\&\&\,\beta>0\,\&\&\,m>0\},\ \text{Solve}\left[\phi==\frac{M\left(\pi+2\,\text{ArcCsc}\left[\sqrt{2}\,r\,\sqrt{\frac{m\,(e-\beta)}{M^2+2\,m\,\alpha}}\right]\right)}{\sqrt{2}\,\sqrt{m}\,\sqrt{\frac{2\,M^2}{m}+4\,\alpha}},\ r\right]\right]$$

$$\left\{\left\{r\to\frac{\text{Csc}\left[\frac{-M\,\pi+2\,\sqrt{m}\,\sqrt{\frac{M^2+2\,m\,\alpha}{m}}\,\phi}{2\,M}\right]}{\sqrt{2}\,\sqrt{\frac{m\,(e-\beta)}{M^2+2\,m\,\alpha}}}\quad\text{if}\quad \textit{condition}\right\}\right\}$$

$$\text{Clear}[r,\ e]$$

$$\sqrt{\frac{m}{2}}\ \text{Assuming}\left[\{\alpha>0\,\&\&\,\beta>0\,\&\&\,m>0\,\&\&\,e>0\},\ \text{Integrate}\left[\frac{1}{\sqrt{e-\left(\frac{\alpha+\beta*q^2}{q^2}+\frac{M^2}{2\,m*q^2}\right)}},\ \left\{q,\ \frac{\sqrt{-M^2-2\,m*\alpha}}{\sqrt{-2\,e*m+2\,m*\beta}},\ r\right\}\right]\right]$$

$$\frac{\sqrt{m}\,r\,\sqrt{e-\frac{M^2+2\,m\,(\alpha+r^2\,\beta)}{2\,m\,r^2}}}{\sqrt{2}\,(e-\beta)}\quad\text{if}\quad \textit{condition}$$

$$\text{Assuming}\left[\{\alpha>0\,\&\&\,\beta>0\,\&\&\,m>0\,\&\&\,e>0\},\ \text{Solve}\left[t==\frac{\sqrt{m}\,r\,\sqrt{e-\frac{M^2+2\,m\,(\alpha+r^2\,\beta)}{2\,m\,r^2}}}{\sqrt{2}\,(e-\beta)},\ r\right]\right]\ //\ \text{FullSimplify}$$

$$\left\{\left\{r\to-\frac{\sqrt{M^2+2\,m\,\alpha+4\,t^2\,(e-\beta)^2}}{\sqrt{2}\,\sqrt{m}\,\sqrt{e-\beta}}\right\},\ \left\{r\to\frac{\sqrt{M^2+2\,m\,\alpha+4\,t^2\,(e-\beta)^2}}{\sqrt{2}\,\sqrt{m}\,\sqrt{e-\beta}}\right\}\right\}$$

$$\text{Clear}[r,\ t];$$

$$r=\frac{\sqrt{M^2+2\,m\,\alpha+4\,t^2\,(e-\beta)^2}}{\sqrt{2}\,\sqrt{m}\,\sqrt{e-\beta}};$$

$$\frac{M\left(\pi+2\,\text{ArcCsc}\left[\sqrt{2}\,r\,\sqrt{\frac{m\,(e-\beta)}{M^2+2\,m\,\alpha}}\right]\right)}{\sqrt{2}\,\sqrt{m}\,\sqrt{\frac{2\,M^2}{m}+4\,\alpha}}\ //\ \text{FullSimplify}$$

$$\frac{M\left(\pi-\text{ArcSec}\left[\frac{\sqrt{M^2+2\,m\,\alpha+4\,t^2\,(e-\beta)^2}\,\sqrt{\frac{m\,(e-\beta)}{M^2+2\,m\,\alpha}}}{\sqrt{m}\,\sqrt{e-\beta}}\right]\right)}{\sqrt{m}\,\sqrt{\frac{M^2}{m}+2\,\alpha}}$$

(ii)

$$\phi_0 = \frac{M}{\sqrt{2m}} \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{E - \left(\frac{\alpha+\beta r^2}{r^2} + \frac{M^2}{2mr^2}\right)}}$$

$$= \frac{M\pi}{2\sqrt{M^2+2m\alpha}}$$

$$\chi = |\pi - 2\phi_0|$$

$$\chi = \pi\left|\frac{M}{\sqrt{M^2+2m\alpha}} - 1\right|$$

$$= \pi\left|\frac{1}{\sqrt{1+\frac{2\alpha}{m\rho^2 {v_\infty}^2}}} - 1\right|$$

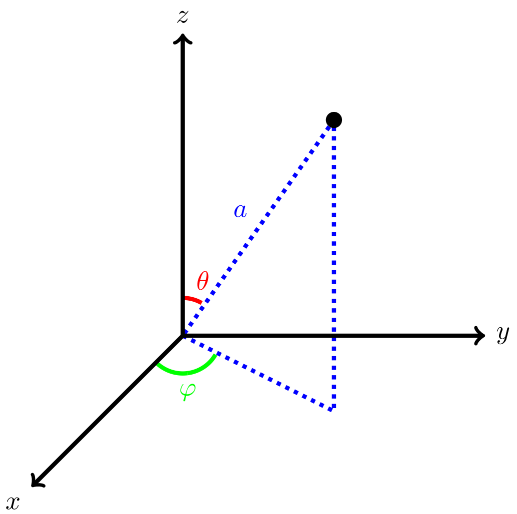
```
M/Sqrt[2 m] Assuming[{α > 0 && β > 0 && m > 0}, Integrate[1/(q^2 Sqrt[e - ((α+β*q^2)/q^2 + M^2/(2 m*q^2))]), {q, Sqrt[-M^2 - 2 m*α]/Sqrt[-2 e*m + 2 m*β], Infinity}]]
```

```
M π/(Sqrt[2] Sqrt[m] Sqrt[2 M^2/m + 4 α])  if  Sqrt[-m (M^2 + 2 m α)] Sqrt[-e + β] > 0
```

Problem 3

(a)

$$x^2 + y^2 + z^2 = a^2$$
$$\implies r = a$$

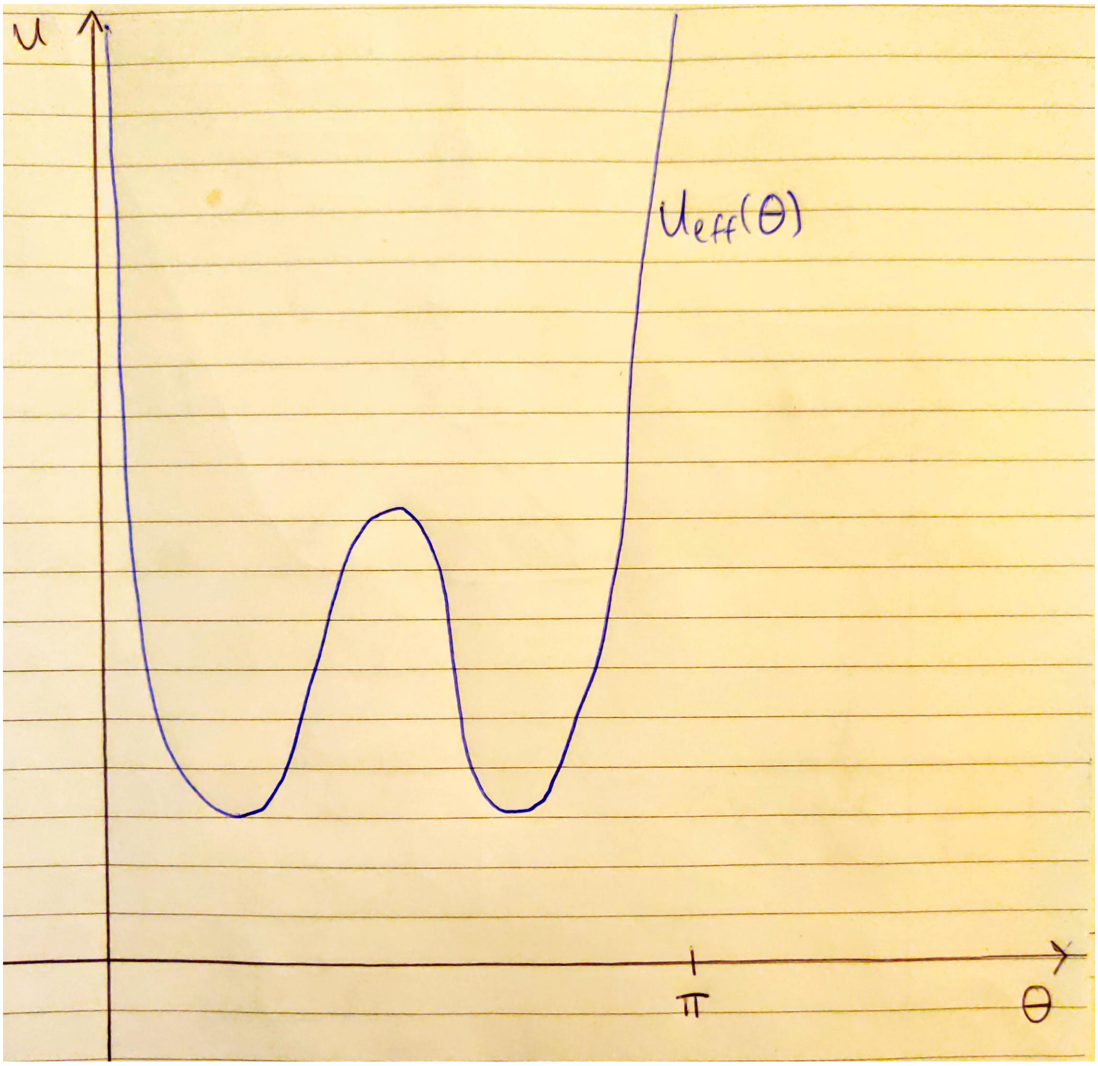


$$x = a\sin\theta\cos\varphi \qquad \dot{x} = a\dot{\theta}\cos\theta\cos\varphi - a\dot{\varphi}\sin\theta\sin\varphi$$
$$y = a\sin\theta\sin\varphi \qquad \dot{y} = a\dot{\theta}\cos\theta\sin\varphi + a\dot{\varphi}\sin\theta\cos\varphi$$
$$z = a\cos\theta \qquad \dot{z} = -a\dot{\theta}\sin\theta$$

$$\mathcal{L}_\lambda = \frac{ma^2}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta - w^2\sin^2\theta\right) + \lambda\left(r^2 - a^2\right)$$
$$\varphi \text{ is cyclic} \implies M = ma^2\sin^2\theta\dot{\varphi} = \text{constant}$$
$$\implies \dot{\varphi}^2 = \frac{M^2}{m^2a^4\sin^4\theta}$$
$$\implies \mathcal{L}_\lambda = \frac{ma^2\dot{\theta}^2}{2} + \frac{M^2}{2ma^2\sin^2\theta} - \frac{mw^2a^2\sin^2\theta}{2} + \lambda\left(r^2 - a^2\right)$$

(b)

$$U(\theta) = \frac{mw^2a^2\sin^2\theta}{2}$$
$$U_{\text{eff}}(\theta) = \frac{mw^2a^2\sin^2\theta}{2} + \frac{M^2}{2ma^2\sin^2\theta}$$



$$U_{\text{eff}}\left(\frac{\pi}{2}\right) = \frac{mw^2a^2}{2} + \frac{M^2}{2ma^2}$$
$$U_{\text{eff}}(\rho) = \frac{mw^2\rho^2}{2} + \frac{M^2}{2m\rho^2}$$
$$\rho_{\text{turning}} = a\sin\theta_{\text{turning}}$$
$$= \sqrt{\frac{M}{mw}}$$
$$U_{\text{eff,min}} = U_{\text{eff}}(\theta_{\text{turning}})$$
$$= Mw$$

If $Mw < E < \frac{mw^2a^2}{2} + \frac{M^2}{2ma^2}$ then the motion is finite. The particle can move on the sphere of radius a, but only between two pairs of values for z.

If $E > \frac{mw^2a^2}{2} + \frac{M^2}{2ma^2}$ then the motion is also finite. The particle can move on the sphere of radius a, but only between two values for z.

$$Mw < E < \frac{mw^2a^2}{2} + \frac{M^2}{2ma^2} \implies \theta_1 = \arcsin\left(\sqrt{\frac{E}{a^2mw^2} - \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

$$\theta_2 = \arcsin\left(\sqrt{\frac{E}{a^2mw^2} + \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

$$\theta_3 = \pi - \arcsin\left(\sqrt{\frac{E}{a^2mw^2} + \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

$$\theta_4 = \pi - \arcsin\left(\sqrt{\frac{E}{a^2mw^2} - \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

$$E > \frac{mw^2a^2}{2} + \frac{M^2}{2ma^2} \implies \theta_1 = \arcsin\left(\sqrt{\frac{E}{a^2mw^2} - \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

$$\theta_2 = \pi - \arcsin\left(\sqrt{\frac{E}{a^2mw^2} - \frac{\sqrt{a^4m^2\left(E^2 - M^2w^2\right)}}{a^4m^2w^2}}\right)$$

```
Clear[e, a, m, w, M, θ, ρ];
Assuming[{m > 0 && w > 0}, Solve[D[m w^2 ρ^2/2 + M^2/(2 m ρ^2), ρ] == 0, ρ]]
```

$$\left\{\left\{\rho \to -\frac{\sqrt{M}}{\sqrt{m}\sqrt{w}}\right\}, \left\{\rho \to -\frac{i\sqrt{M}}{\sqrt{m}\sqrt{w}}\right\}, \left\{\rho \to \frac{i\sqrt{M}}{\sqrt{m}\sqrt{w}}\right\}, \left\{\rho \to \frac{\sqrt{M}}{\sqrt{m}\sqrt{w}}\right\}\right\}$$

```
Clear[e, a, m, w, M, θ, ρ];
Assuming[{m > 0 && w > 0 && a > 0 && 0 < θ < π}, Solve[e == m w^2 a^2 (Sin[θ])^2/2 + M^2/(2 m a^2 (Sin[θ])^2), θ]]
```

$$\left\{\left\{\theta \to -\text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} - \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\theta \to \pi - \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} - \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\},\right.$$

$$\left\{\theta \to \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} - \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\theta \to \pi + \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} - \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\},$$

$$\left\{\theta \to -\text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} + \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\theta \to \pi - \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} + \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\},$$

$$\left.\left\{\theta \to \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} + \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\theta \to \pi + \text{ArcSin}\left[\sqrt{\frac{e}{a^2 m w^2} + \frac{\sqrt{a^4 m^2 (e^2 - M^2 w^2)}}{a^4 m^2 w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$$

(c)

$$
\begin{aligned}
E &= U_{\text{eff,min}} \\
&= Mw \\
2m\pi ab &= TM \text{ for an ellipse} \\
\implies T &= \frac{2m\pi R^2}{M} \\
R &= \rho_{\text{turning}} \\
&= \sqrt{\frac{M}{mw}} \\
\implies T &= \frac{2\pi}{w}
\end{aligned}
$$

$$
\begin{aligned}
E &= Mw \\
&= \frac{m\omega^2}{2}\frac{M}{mw} \\
\implies w &= \frac{\omega}{\sqrt{2}} \\
\implies T &= \frac{2\sqrt{2}\pi}{\omega}
\end{aligned}
$$

For small ω, T tends to infinity. This makes sense as the time it takes for the particle to travel in a circle will increase as the angular speed of the particle decreases.

For large ω, T tends to 0. This makes sense as the time it takes for the particle to travel in a circle will decrease as the angular speed of the particle increases.