MAU23401: Advanced Classical Mechanics I
Homework 4 due 30/10/2020

Ruaidhrí Campion
19333850
SF Theoretical Physics

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Problem 1

1.

Maximising $U(x)$ is the same as finding the turning point(s) of $e^{ax} + \frac{1}{4}e^{-ax}$.

$$\begin{aligned}\frac{d}{dt}\left(e^{ax} + \frac{1}{4}e^{-ax}\right) = ae^{ax} - \frac{a}{4}e^{-ax} &= 0\\ e^{ax} - \frac{1}{4}e^{-ax} &= 0\\ e^{2ax} &= \frac{1}{4}\\ 2ax &= \ln\frac{1}{4}\\ x &= \frac{\ln\frac{1}{4}}{2a}\end{aligned}$$

$$\begin{aligned}U\left(\frac{\ln\frac{1}{4}}{2a}\right) &= \frac{V}{\left(e^{\frac{a\ln\frac{1}{4}}{2a}} + \frac{1}{4}e^{\frac{a\ln 4}{2a}}\right)^2}\\ &= \frac{V}{\left(\frac{1}{2} + \frac{1}{2}\right)^2}\\ U_{\text{max}} &= V\end{aligned}$$

2.

Change of coordinates

$$
\begin{aligned}
y &= x + \frac{\ln 4}{2a} \\
\implies x &= y - \frac{\ln 4}{2a} \\
U(y) &= \frac{V}{\left(e^{ay}e^{\frac{\ln \frac{1}{4}}{2}} + \frac{1}{4}e^{-ay}e^{\frac{\ln 4}{2}}\right)^2} \\
&= \frac{V}{\left(\frac{1}{2}e^{ay} + \frac{1}{2}e^{-ay}\right)^2} \\
&= \frac{V}{\cosh^2(ay)}
\end{aligned}
$$

Change of coordinates

$$
\begin{aligned}
z &= \sinh(ay) \\
\implies y &= \frac{1}{a}\sinh^{-1} z \\
U(z) &= \frac{V}{\cosh^2\left(\sinh^{-1} z\right)} \\
&= \frac{V}{\sinh^2\left(\sinh^{-1} z\right) + 1} \\
&= \frac{V}{z^2 + 1} \\
K &= \frac{m}{2}\left(\frac{dx}{dt}\right)^2 \\
&= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 \\
&= \frac{m}{2}\left(\frac{dy}{dz}\frac{dz}{dt}\right)^2 \\
&= \frac{m}{2a^2}\frac{1}{z^2+1}\left(\frac{dz}{dt}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
E &= K + U \\
&= \frac{m}{2a^2}\frac{1}{z^2+1}\left(\frac{dz}{dt}\right)^2 + \frac{V}{z^2+1} \\
\frac{dz}{dt} &= a\sqrt{\frac{2\left(E\left(1+z^2\right)-V\right)}{m}} \\
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{\sqrt{E\left(1+z^2\right)-V}} \\
T_L(U) &= \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dz}{\sqrt{E\left(1+z^2\right)-V}} \\
T_L(0) &= \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dz}{\sqrt{E\left(1+z^2\right)}} \\
T_{\text{delay}} &= \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{E\left(1+z^2\right)-V}} - \frac{1}{\sqrt{E\left(1+z^2\right)}}\right)dz \\
&= \frac{\sqrt{2m}}{a}\int_{0}^{\infty}\left(\frac{1}{\sqrt{E\left(1+z^2\right)-V}} - \frac{1}{\sqrt{E\left(1+z^2\right)}}\right)dz \\
T_{\text{delay}} &= \frac{1}{a}\sqrt{\frac{m}{2E}}\ln\left(\frac{E}{E-V}\right)
\end{aligned}
$$

$$\frac{\sqrt{2\,m}}{a}\ \text{Assuming}\left[\{e > V,\ V > 0\},\ \text{Integrate}\left[\frac{1}{\sqrt{e * \left(1+z^2\right) - V}} - \frac{1}{\sqrt{e * \left(1+z^2\right)}},\ \{z,\ 0,\ \text{Infinity}\}\right]\right]$$

$$\frac{\sqrt{m}\ \text{Log}\left[\frac{e}{e-V}\right]}{\sqrt{2}\ a\ \sqrt{e}}$$

3.

$$E \approx V \implies T_{\text{delay}} = \frac{1}{a}\sqrt{\frac{m}{2V}}\ln\left(\frac{V}{E-V}\right)$$

$\approx \infty$ as the motion is almost bounded

$$E >> V \implies T_{\text{delay}} = \frac{1}{a}\sqrt{\frac{m}{2}}\frac{V}{E^{\frac{3}{2}}}$$

≈ 0 as the motion is very close to free motion

```
Tdelay = (Sqrt[m] Log[e/(e-V)])/(Sqrt[2] a Sqrt[e]);
Series[Tdelay, {e, V, 0}] // Normal // Expand
Series[Tdelay, {e, Infinity, 2}] // Normal // Expand
```

$$\frac{\sqrt{m}\,\text{Log}\left[\frac{V}{e-V}\right]}{\sqrt{2}\,a\,\sqrt{V}}$$

$$\frac{\sqrt{m}\,V}{\sqrt{2}\,a\,e^{3/2}}$$

4.

$$\begin{aligned}
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{\sqrt{E\left(1+z^2\right)-V}} \\
t &= \frac{1}{a}\sqrt{\frac{m}{2E}}\sinh^{-1}\left(z\sqrt{\frac{E}{E-V}}\right) \\
z &= \sqrt{\frac{E-V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right) \\
y &= \frac{1}{a}\sinh^{-1} z \\
&= \frac{1}{a}\sinh^{-1}\left(\sqrt{\frac{E-V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right)\right) \\
x &= y - \frac{\ln 4}{2a} \\
x(t) &= \frac{1}{a}\sinh^{-1}\left(\sqrt{\frac{E-V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right)\right) - \frac{\ln 4}{2a}
\end{aligned}$$

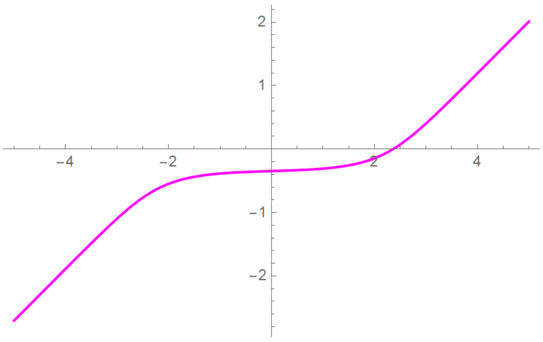
$$\frac{1}{a}\sqrt{\frac{m}{2}}\ \text{Assuming}\left[\{e>V,\ V>0\},\ \text{Integrate}\left[\frac{1}{\sqrt{e\left(1+q^2\right)-V}},\ \{q,\ 0,\ z\}\right]\right]$$

$$\frac{\sqrt{m}\ \text{ArcSinh}\left[\sqrt{\frac{e}{e-V}}\ z\right]}{\sqrt{2}\ a\ \sqrt{e}}\quad \text{if}\ \ \text{Re}[z]>0\ \&\&\ \text{Im}[z]==0$$

5.

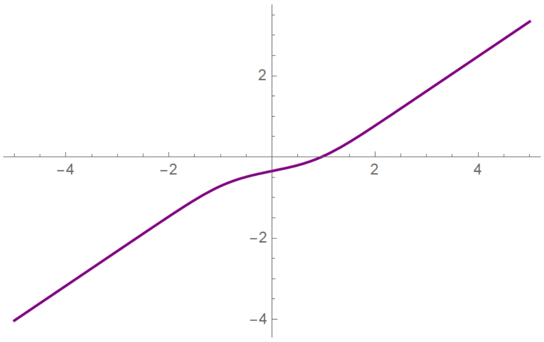
$$V = 1 \qquad a = 2 \qquad m = 3$$

```
Plot[1/2 ArcSinh[Sqrt[(1.001 - 1)/1.001] Sinh[2 * t Sqrt[(2 * 1.001)/3]]] - Log[4]/(2 * 2), {t, -5, 5}, PlotStyle → {Magenta, Thick}]
```



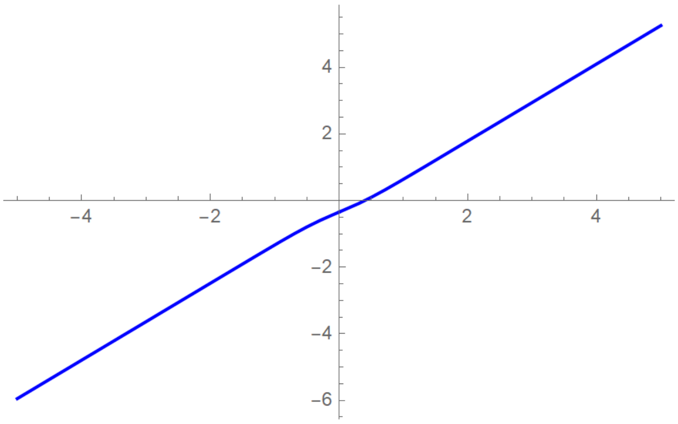
$$E = 1.001$$

```
Plot[1/2 ArcSinh[Sqrt[(1.1 - 1)/1.1] Sinh[2 * t Sqrt[(2 * 1.1)/3]]] - Log[4]/(2 * 2), {t, -5, 5}, PlotStyle → {Purple, Thick}]
```



$$E = 1.1$$

```
Plot[1/2 ArcSinh[Sqrt[(2 - 1)/2] Sinh[2 * t Sqrt[(2 * 2)/3]]] - Log[4]/(2 * 2), {t, -5, 5}, PlotStyle → {Blue, Thick}]
```



$$E = 2$$

Problem 2

1.

Minimising $U(x)$ is the same as finding the turning point(s) of $4e^{-2ax} + \frac{1}{4}e^{2ax} + 2$.

$$\begin{aligned}\frac{d}{dt}\left(4e^{-2ax} + \frac{1}{4}e^{2ax} + 2\right) = -8ae^{-2ax} + \frac{a}{2}e^{2ax} &= 0 \\ e^{4ax} &= 16 \\ x &= \frac{\ln 4}{2a}\end{aligned}$$

$$\begin{aligned}U\left(\frac{\ln 4}{2a}\right) &= -\frac{4V}{4\left(\frac{1}{4}\right) + \frac{1}{4} + 2} \\ U_{\text{min}} &= -V\end{aligned}$$

2.

(a)

$$\lim_{x\to-\infty} U(x) = \lim_{x\to\infty} u(x) = 0$$

$$\begin{aligned}\text{Finite motion} &\implies E < 0 \\ \text{Infinite motion} &\implies E \geqslant 0\end{aligned}$$

(b)

Change of coordinates

$$\begin{aligned}
y &= x - \frac{\ln 4}{2a} \\
\implies x &= y + \frac{\ln 4}{2a} \\
U(y) &= -\frac{4V}{4e^{-2ay}e^{-\ln 4} + \frac{1}{4}e^{2ay}e^{\ln 4} + 2} \\
&= -\frac{4V}{e^{2ay} + 2 + e^{-2ay}} \\
&= -\frac{V}{\cosh^2(ay)}
\end{aligned}$$

Change of coordinates

$$\begin{aligned}
z &= \sinh(ay) \\
\implies y &= \frac{1}{a}\sinh^{-1} z \\
U(z) &= -\frac{V}{\cosh^2\left(\sinh^{-1} z\right)} \\
&= -\frac{V}{\sinh^2\left(\sinh^{-1} z\right) + 1} \\
&= -\frac{V}{z^2 + 1} \\
K &= \frac{m}{2}\left(\frac{dx}{dt}\right)^2 \\
&= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 \\
&= \frac{m}{2}\left(\frac{dy}{dz}\frac{dz}{dt}\right)^2 \\
&= \frac{m}{2a^2}\frac{1}{z^2 + 1}\left(\frac{dz}{dt}\right)^2
\end{aligned}$$

$$
\begin{aligned}
E &= K + U \\
&= \frac{m}{2a^2}\frac{1}{z^2+1}\left(\frac{dz}{dt}\right)^2 - \frac{V}{z^2+1} \\
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{\sqrt{E\left(z^2+1\right)+V}} \\
U(z) &= -\frac{V}{z^2+1} = E \\
z &= \pm\sqrt{\frac{-E-V}{E}} \\
\implies T &= \frac{2}{a}\sqrt{\frac{m}{2}}\int_{-\sqrt{\frac{-E-V}{E}}}^{\sqrt{\frac{-E-V}{E}}}\frac{dz}{\sqrt{E\left(z^2+1\right)+V}} \\
&= \frac{2\sqrt{2m}}{a}\int_0^{\sqrt{\frac{-E-V}{E}}}\frac{dz}{\sqrt{E\left(z^2+1\right)+V}} \\
T &= \frac{\pi}{a}\sqrt{-\frac{2m}{E}}
\end{aligned}
$$

$$\frac{2\sqrt{2\,m}}{a}\,\texttt{Assuming}\left[\{e<0\,\&\&\,V>0\,\&\&\,-e-V<0\},\ \texttt{Integrate}\left[\frac{1}{\sqrt{e\left(z^2+1\right)+V}},\ \left\{z,\ 0,\ \sqrt{\frac{-e-V}{e}}\right\}\right]\right]$$

$$\frac{\sqrt{2}\,\sqrt{m}\,\pi}{a\,\sqrt{-e}}$$

(c)

$$
\begin{aligned}
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{E\left(z^2+1\right)+V} \\
t &= \frac{1}{a}\sqrt{-\frac{m}{2E}}\sin^{-1}\left(z\sqrt{-\frac{E}{E+V}}\right) \\
z &= \sqrt{-\frac{E+V}{E}}\sin\left(at\sqrt{-\frac{2E}{m}}\right) \\
y &= \frac{1}{a}\sinh^{-1} z \\
&= \frac{1}{a}\sinh^{-1}\left(\sqrt{-\frac{E+V}{E}}\sin\left(at\sqrt{-\frac{2E}{m}}\right)\right) \\
x &= y + \frac{\ln 4}{2a} \\
x(t) &= \frac{1}{a}\sinh^{-1}\left(\sqrt{-\frac{E+V}{E}}\sin\left(at\sqrt{-\frac{2E}{m}}\right)\right) + \frac{\ln 4}{2a}
\end{aligned}
$$

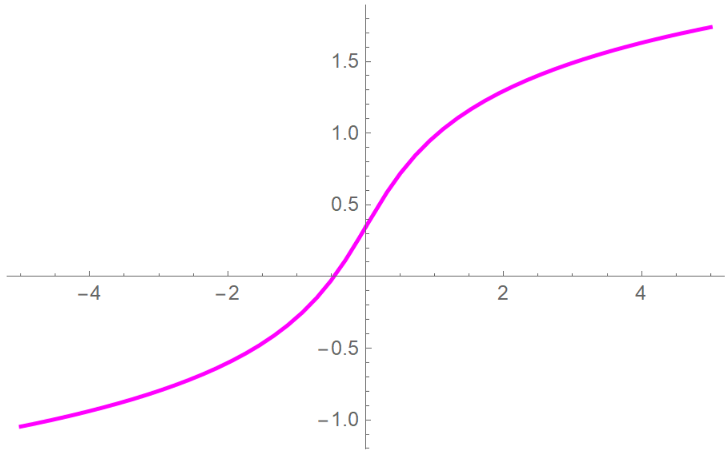
$$\frac{1}{a}\sqrt{\frac{m}{2}}\ \texttt{Assuming}\left[\{e<0\,\&\&\,V>0\},\ \texttt{Integrate}\left[\frac{1}{\sqrt{e\left(q^2+1\right)+V}},\ \{q,0,z\}\right]\right]$$

$$\frac{\sqrt{m}\ \texttt{ArcSin}\left[\sqrt{-\frac{e}{e+V}}\ z\right]}{\sqrt{2}\ a\ \sqrt{-e}}\quad \texttt{if}\quad z>0\,\&\&\,e+V+e\,z^2>0$$

(d)

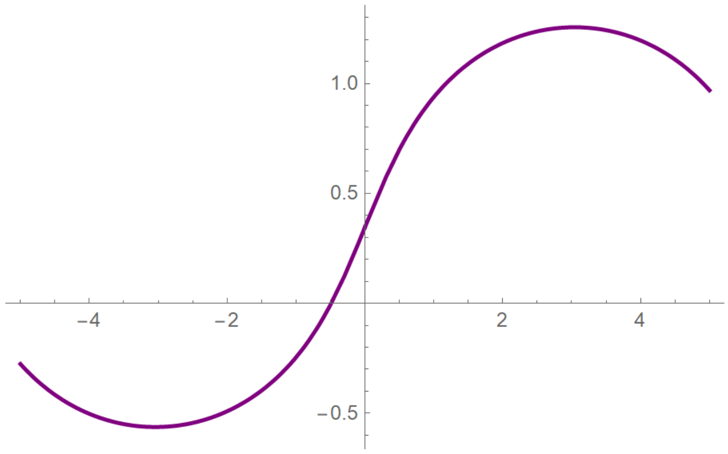
$V = 1 \qquad a = 2 \qquad m = 3$

$$\text{Plot}\left[\frac{1}{2}\,\text{ArcSinh}\left[\sqrt{-\frac{-0.001+1}{-0.001}}\ \text{Sin}\left[2*t\sqrt{-\frac{2*-0.001}{3}}\right]\right]+\frac{\text{Log}[4]}{2*2},\ \{t,-5,5\},\ \text{PlotStyle}\to\{\text{Magenta},\ \text{Thick}\}\right]$$



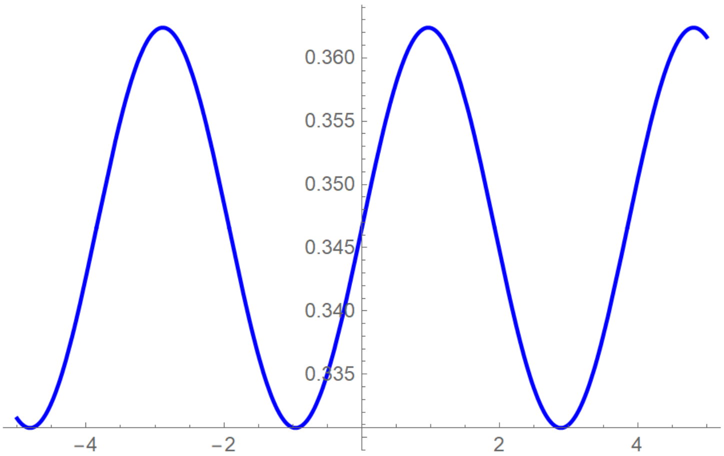
$E = -0.001$

$$\text{Plot}\left[\frac{1}{2}\,\text{ArcSinh}\left[\sqrt{-\frac{-0.1+1}{-0.1}}\ \text{Sin}\left[2*t\sqrt{-\frac{2*-0.1}{3}}\right]\right]+\frac{\text{Log}[4]}{2*2},\ \{t,-5,5\},\ \text{PlotStyle}\to\{\text{Purple},\ \text{Thick}\}\right]$$



$E = -0.1$

```
Plot[1/2 ArcSinh[Sqrt[-(-0.999 + 1)/-0.999] Sin[2 * t Sqrt[-(2 * -0.999)/3]]] + Log[4]/(2 * 2), {t, -5, 5},
 PlotStyle → {Blue, Thick}]
```



$$E = -0.999$$

3.

(a)

$$dt = \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{E\left(z^2+1\right)+V}$$

$$T_L(U) = \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dz}{E\left(z^2+1\right)+V}$$

$$T_L(0) = \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dz}{E\left(z^2+1\right)}$$

$$T_{\text{delay}} = \frac{1}{a}\sqrt{\frac{m}{2}}\int_{-\infty}^{\infty}\left(\frac{1}{E\left(z^2+1\right)+V}-\frac{1}{E\left(z^2+1\right)}\right)dz$$

$$= \frac{\sqrt{2m}}{a}\int_{0}^{\infty}\left(\frac{1}{E\left(z^2+1\right)+V}-\frac{1}{E\left(z^2+1\right)}\right)dz$$

$$T_{\text{delay}} = \frac{1}{a}\sqrt{\frac{m}{2E}}\ln\left(\frac{E}{E+V}\right)$$

```
Sqrt[2 m]/a Assuming[{e ≥ 0 && V > 0}, Integrate[1/Sqrt[e (z^2 + 1) + V] - 1/Sqrt[e (z^2 + 1)], {z, 0, Infinity}]]

(Sqrt[m] Log[e/(e+V)])/(Sqrt[2] a Sqrt[e])
```

(b)

$$E \approx 0 \implies T_{\text{delay}} = \frac{1}{a}\sqrt{\frac{m}{2E}}\ln\left(\frac{E}{V}\right)$$

$$\approx -\infty \text{ as the motion is almost bounded}$$

$$E >> V \implies T_{\text{delay}} = -\frac{1}{a}\sqrt{\frac{m}{2}}\frac{V}{E^{\frac{3}{2}}}$$

$$\approx 0 \text{ as the motion is very close to free motion}$$

```
Tdelay = (Sqrt[m] Log[e/(e+V)])/(Sqrt[2] a Sqrt[e]);
Series[Tdelay, {e, 0, 0}] // Normal // Expand
Series[Tdelay, {e, Infinity, 2}] // Normal // Expand
```

$$\frac{\sqrt{m}\,\text{Log}\left[\frac{e}{V}\right]}{\sqrt{2}\,a\,\sqrt{e}}$$

$$-\frac{\sqrt{m}\,V}{\sqrt{2}\,a\,e^{3/2}}$$

(c)

$$dt = \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{E\left(z^2+1\right)+V}$$

$$t = \frac{1}{a}\sqrt{\frac{m}{2E}}\sinh^{-1}\left(z\sqrt{\frac{E}{E+V}}\right)$$

$$z = \sqrt{\frac{E+V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right)$$

$$y = \frac{1}{a}\sinh^{-1} z$$

$$= \frac{1}{a}\sinh^{-1}\left(\sqrt{\frac{E+V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right)\right)$$

$$x = y + \frac{\ln 4}{2a}$$

$$x(t) = \frac{1}{a}\sinh^{-1}\left(\sqrt{\frac{E+V}{E}}\sinh\left(at\sqrt{\frac{2E}{m}}\right)\right) + \frac{\ln 4}{2a}$$

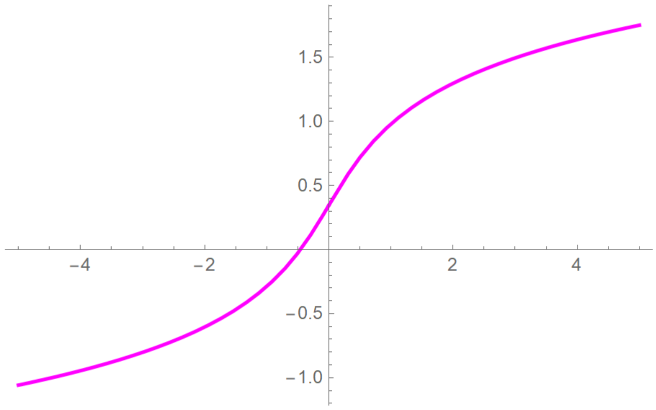
```
1/a Sqrt[m/2] Assuming[{e ≥ 0 && V > 0}, Integrate[1/Sqrt[e (q^2 + 1) + V], {q, 0, z}]]
```

$$\frac{\sqrt{m}\,\text{ArcSinh}\left[\sqrt{\frac{e}{e+V}}\;z\right]}{\sqrt{2}\,a\,\sqrt{e}} \quad \text{if } e > 0 \,\&\&\, \text{Re}[z] > 0 \,\&\&\, \text{Im}[z] == 0$$

(d)

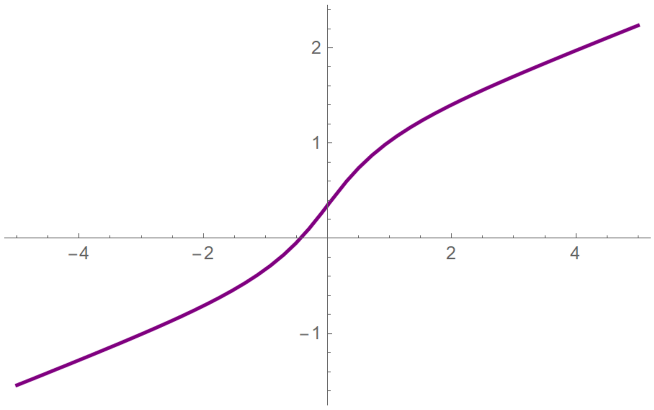
$$V = 1 \qquad a = 2 \qquad m = 3$$

```
Plot[1/2 ArcSinh[Sqrt[(0.001 + 1)/0.001] Sinh[2 * t Sqrt[(2 * 0.001)/3]]] + Log[4]/(2 * 2), {t, -5, 5},
 PlotStyle → {Magenta, Thick}]
```



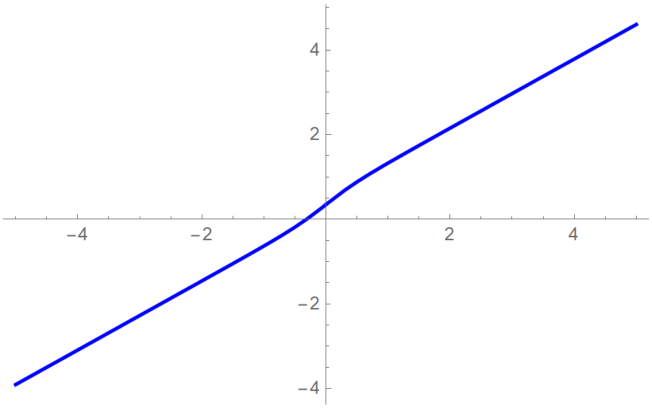
$$E = 0.001$$

```
Plot[1/2 ArcSinh[Sqrt[(0.1 + 1)/0.1] Sinh[2 * t Sqrt[(2 * 0.1)/3]]] + Log[4]/(2 * 2), {t, -5, 5},
 PlotStyle → {Purple, Thick}]
```



$$E = 0.1$$

```
Plot[1/2 ArcSinh[Sqrt[(1 + 1)/1] Sinh[2 * t Sqrt[(2 * 1)/3]]] + Log[4]/(2 * 2), {t, -5, 5}, PlotStyle → {Blue, Thick}]
```



$$E = 1$$

Problem 3

1.

$$\begin{aligned}\frac{dU}{dx} &= -2aV\cot(ax)\csc^2(ax)\\ &= 0\\ x &= \frac{\pi}{2a}\end{aligned}$$

Change of coordinates

$$\begin{aligned} y &= x - \frac{\pi}{2a}\\ \implies x &= y + \frac{\pi}{2a}\\ U(y) &= V\cot^2\left(ay - \frac{\pi}{2}\right)\\ &= V\tan^2(ay)\end{aligned}$$

Change of coordinates

$$\begin{aligned} z &= \sin(ay)\\ \implies y &= \frac{1}{a}\sin^{-1} z\\ U(z) &= V\tan^2\left(\sin^{-1} z\right)\\ &= V\left(\sec^2\left(\sin^{-1} z\right) - 1\right)\\ &= V\left(\frac{1}{\cos^2\left(\sin^{-1} z\right)} - 1\right)\\ &= V\left(\frac{1}{1-z^2} - 1\right)\\ &= \frac{Vz^2}{1-z^2}\\ K &= \frac{m}{2}\left(\frac{dx}{dt}\right)^2\\ &= \frac{m}{2}\left(\frac{dy}{dt}\right)^2\\ &= \frac{m}{2}\left(\frac{dy}{dz}\frac{dz}{dt}\right)^2\\ &= \frac{m}{2a^2}\frac{1}{1-z^2}\left(\frac{dz}{dt}\right)^2\end{aligned}$$

$$\begin{aligned}
E &= K + U \\
&= \frac{m}{2a^2}\frac{1}{1-z^2}\left(\frac{dz}{dt}\right)^2 + \frac{Vz^2}{1-z^2} \\
\frac{dz}{dt} &= a\sqrt{\frac{2\left(E\left(1-z^2\right) - Vz^2\right)}{m}} \\
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{\sqrt{E\left(1-z^2\right) - Vz^2}} \\
U(z) &= \frac{Vz^2}{1-z^2} = E \\
z &= \pm\sqrt{\frac{E}{E+V}} \\
T &= \frac{\sqrt{2m}}{a}\int_{-\sqrt{\frac{E}{E+V}}}^{\sqrt{\frac{E}{E+V}}}\frac{dz}{\sqrt{E\left(1-z^2\right) - Vz^2}} \\
&= \frac{2\sqrt{2m}}{a}\int_0^{\sqrt{\frac{E}{E+V}}}\frac{dz}{\sqrt{E\left(1-z^2\right) - Vz^2}} \\
T &= \frac{\pi}{a}\sqrt{\frac{2m}{E+V}}
\end{aligned}$$

$$\frac{2\sqrt{2\,m}}{a}\ \text{Assuming}\left[\left\{e > 0\ \&\&\ V > 0\ \&\&\ e\left(1 - z^2\right) - V * z^2 > 0\right\},\ \text{Integrate}\left[\frac{1}{\sqrt{e\left(1 - z^2\right) - V * z^2}}, \left\{z, 0, \sqrt{\frac{e}{e + V}}\right\}\right]\right]$$

$$\frac{\sqrt{2}\ \sqrt{m}\ \pi}{a\ \sqrt{e + V}}$$

2.

$$E \approx 0 \implies T = \frac{\pi}{a}\sqrt{\frac{2m}{V}} \text{ as the motion is almost harmonic}$$

$$E >> V \implies T = \frac{\pi}{a}\sqrt{\frac{2m}{E}}$$
$$\approx 0 \text{ as the motion is almost unbounded}$$

$$T = \frac{\sqrt{2}\,\sqrt{m}\,\pi}{a\,\sqrt{e+V}};$$

```
Series[T, {e, 0, 0}] // Normal // Expand
Series[T, {e, Infinity, 1}] // Normal // Expand
```

$$\frac{\sqrt{2}\,\sqrt{m}\,\pi}{a\,\sqrt{V}}$$

$$\frac{\sqrt{2}\,\sqrt{m}\,\pi}{a\,\sqrt{e}}$$

3.

$$
\begin{aligned}
dt &= \frac{1}{a}\sqrt{\frac{m}{2}}\frac{dz}{\sqrt{E\left(1-z^2\right)-Vz^2}}\\
t &= \frac{1}{a}\sqrt{\frac{m}{2(E+V)}}\sin^{-1}\left(z\sqrt{\frac{E+V}{E}}\right)\\
z &= \sqrt{\frac{E}{E+V}}\sin\left(at\sqrt{\frac{2(E+V)}{m}}\right)\\
y &= \frac{1}{a}\sin^{-1}z\\
&= \frac{1}{a}\sin^{-1}\left(\sqrt{\frac{E}{E+V}}\sin\left(at\sqrt{\frac{2(E+V)}{m}}\right)\right)\\
x &= y+\frac{\pi}{2a}\\
x(t) &= \frac{1}{a}\sin^{-1}\left(\sqrt{\frac{E}{E+V}}\sin\left(at\sqrt{\frac{2(E+V)}{m}}\right)\right)+\frac{\pi}{2a}
\end{aligned}
$$

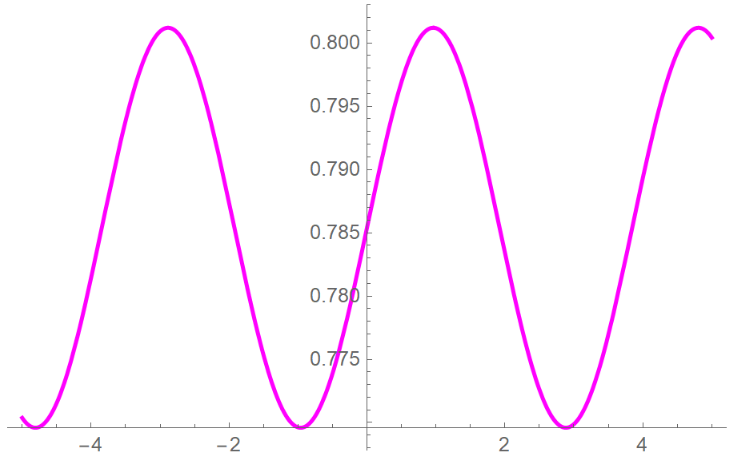
$$\frac{1}{a}\sqrt{\frac{m}{2}}\ \texttt{Assuming}\left[\{e>0\ \&\&\ V>0\},\ \texttt{Integrate}\left[\frac{1}{\sqrt{e\left(1-q^2\right)-V*q^2}},\ \{q,0,z\}\right]\right]$$

$$\frac{\sqrt{m}\ \text{ArcSin}\left[\sqrt{\frac{e+V}{e}}\ z\right]}{\sqrt{2}\ a\ \sqrt{e+V}}\quad \text{if}\quad (e+V)\ z^2 < e\ \&\&\ z>0$$

4.

$V = 1 \qquad a = 2 \qquad m = 3$

```
Plot[1/2 ArcSin[Sqrt[0.001/(0.001 + 1)] Sin[2 * t Sqrt[(2 * (0.001 + 1))/3]]] + π/(2 * 2), {t, -5, 5},
 PlotStyle → {Magenta, Thick}]
```



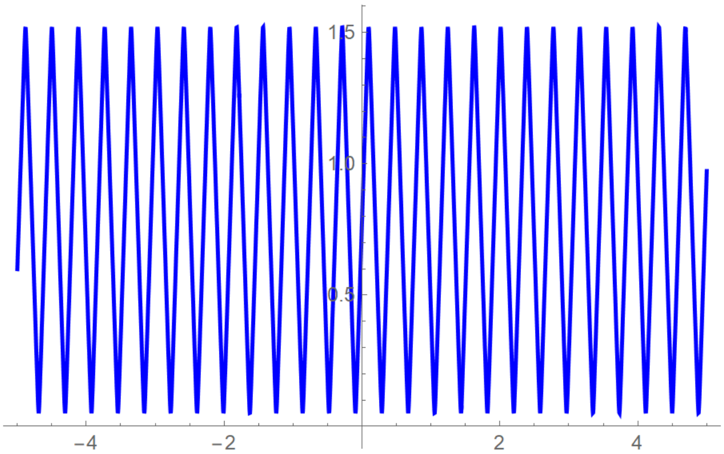
$E = 0.001$

```
Plot[1/2 ArcSin[Sqrt[1/(1 + 1)] Sin[2 * t Sqrt[(2 * (1 + 1))/3]]] + π/(2 * 2), {t, -5, 5}, PlotStyle → {Purple, Thick}]
```

$E = 1$

$$\texttt{Plot}\left[\frac{1}{2}\,\texttt{ArcSin}\left[\sqrt{\frac{100}{100+1}}\;\texttt{Sin}\left[2 * \texttt{t}\,\sqrt{\frac{2*(100+1)}{3}}\right]\right]+\frac{\pi}{2*2},\ \{\texttt{t},\,-5,\,5\},$$

$$\texttt{PlotStyle} \rightarrow \{\texttt{Blue},\ \texttt{Thick}\}\right]$$



$E = 100$