MAU23401: Advanced Classical Mechanics I
Homework 3 due 23/10/2020

Ruaidhrí Campion
19333850
SF Theoretical Physics

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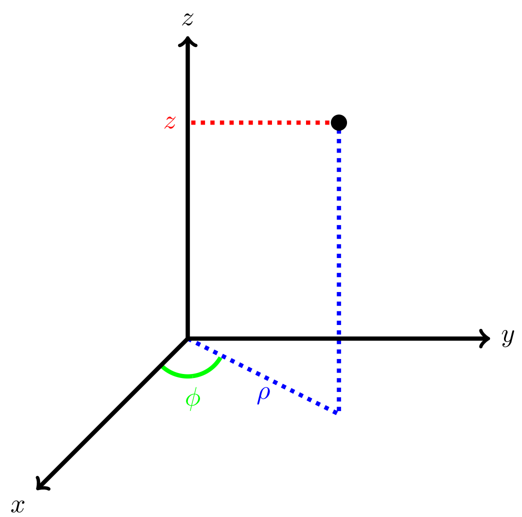
Problem 1

(a)

$$
\begin{aligned}
x &= \rho\cos\phi & \dot{x} &= \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi \\
y &= \rho\sin\phi & \dot{y} &= \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi \\
z &= z & \dot{z} &= \dot{z}
\end{aligned}
$$



$$
\begin{aligned}
L = &\frac{m}{2}\left(\dot{\rho}^2\cos^2\phi + \rho^2\dot{\phi}^2\sin^2\phi - 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + \dot{\rho}^2\sin^2\phi + \rho^2\dot{\phi}^2\cos^2\phi + 2\rho\dot{\rho}\dot{\phi}\cos\phi\sin\phi + \dot{z}^2\right) \\
&+ \beta\left(\rho\cos\phi\sin\frac{z}{\kappa} - \rho\sin\phi\cos\frac{z}{\kappa}\right) \\
L = &\frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos\phi\sin\frac{z}{\kappa} - \sin\phi\cos\frac{z}{\kappa}\right)
\end{aligned}
$$

(b)

$$\rho \to \tilde{\rho} = \rho + \epsilon \implies \begin{cases} L(\rho, \phi, z) \to L(\tilde{\rho}, \phi, z) \\ \dot{\rho} \to \dot{\tilde{\rho}} = \dot{\rho} \end{cases}$$

$$\begin{aligned} L(\tilde{\rho}, \phi, z) &= \frac{m}{2}\left(\dot{\rho}^2 + \dot{\phi}^2\left(\rho^2 + \epsilon^2 + 2\rho\epsilon\right) + \dot{z}^2\right) + \beta(\rho+\epsilon)\left(\cos\phi \sin\frac{z}{\kappa} - \sin\phi\cos\frac{z}{\kappa}\right) \\ &= L(\rho, \phi, z) + \frac{m\dot{\phi}^2}{2}\left(\epsilon^2 + 2\rho\epsilon\right) + \beta\epsilon\left(\cos\phi\sin\frac{z}{\kappa} - \sin\phi\cos\frac{z}{\kappa}\right) \end{aligned}$$

$L(\tilde{\rho}, \phi, z)$ does not differ by a total time derivative, as it contains a $\dot{\phi}^2$. Integrating this will result in a $\dot{\phi}$ term, and thus will depend on $\dot{\phi}$. Thus $\rho \to \tilde{\rho} = \rho + \epsilon$ is not a continuous symmetry.

$$\phi \to \tilde{\phi} = \phi + \epsilon \implies \begin{cases} L(\rho, \phi, z) \to L\left(\rho, \tilde{\phi}, z\right) \\ \dot{\phi} \to \dot{\tilde{\phi}} = \dot{\phi} \end{cases}$$

$$\begin{aligned} L\left(\rho, \tilde{\phi}, z\right) &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos(\phi+\epsilon)\sin\frac{z}{\kappa} - \sin(\phi+\epsilon)\cos\frac{z}{\kappa}\right) \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left((\cos\phi\cos\epsilon - \sin\phi\sin\epsilon)\sin\frac{z}{\kappa} - (\sin\phi\cos\epsilon + \cos\phi\sin\epsilon)\cos\frac{z}{\kappa}\right) \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos\phi\sin\frac{z}{\kappa} - \epsilon\sin\phi\sin\frac{z}{\kappa} - \sin\phi\cos\frac{z}{\kappa} - \epsilon\cos\phi\cos\frac{z}{\kappa}\right) \\ &= L(\rho, \phi, z) - \epsilon\beta\rho\left(\sin\phi\sin\frac{z}{\kappa} + \cos\phi\cos\frac{z}{\kappa}\right) \\ \text{Let } L'(\rho, \phi, z) &= \rho\left(\sin\phi\sin\frac{z}{\kappa} + \cos\phi\cos\frac{z}{\kappa}\right) \\ \frac{d}{dt}\frac{\partial L'}{\partial\dot{\rho}} &= \frac{\partial L'}{\partial\rho} \\ 0 &= \sin\phi\sin\frac{z}{\kappa} + \cos\phi\cos\frac{z}{\kappa} \\ \implies L\left(\rho, \tilde{\phi}, z\right) &= L(\rho, \phi, z) \end{aligned}$$

L is invariant under an infinitesimal change of ϕ, thus $\phi \to \tilde{\phi} = \phi + \epsilon$ is a continuous symmetry.

$$z \to \tilde{z} = z + \kappa\epsilon \implies \begin{cases} L(\rho, \phi, z) \to L(\rho, \phi, \tilde{z}) \\ \dot{z} \to \dot{\tilde{z}} = \dot{z} \end{cases}$$

$$\begin{aligned} L(\rho, \phi, \tilde{z}) &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos\phi\sin\left(\frac{z}{\kappa} + \epsilon\right) - \sin\phi\cos\left(\frac{z}{\kappa} + \epsilon\right)\right) \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos\phi\left(\sin\frac{z}{\kappa}\cos\epsilon + \cos\frac{z}{\kappa}\sin\epsilon\right) - \sin\phi\left(\cos\frac{z}{\kappa}\cos\epsilon - \sin\frac{z}{\kappa}\sin\epsilon\right)\right) \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + \beta\rho\left(\cos\phi\sin\frac{z}{\kappa} + \epsilon\cos\phi\cos\frac{z}{\kappa} - \sin\phi\cos\frac{z}{\kappa} + \epsilon\sin\phi\sin\frac{z}{\kappa}\right) \\ &= L(\rho, \phi, z) + \epsilon\beta\rho\left(\cos\phi\cos\frac{z}{\kappa} + \sin\phi\sin\frac{z}{\kappa}\right) \\ &= L(\rho, \phi, z) \text{ similarly from before.} \end{aligned}$$

L is invariant under an infinitesimal change of z, thus $z \to \tilde{z} = z + \epsilon$ is a continuous symmetry.

$$J = \frac{\partial L}{\partial\dot{q}^i}\zeta^i - \Lambda$$

$$
\begin{aligned}
J_\phi &= \frac{\partial L}{\partial \dot\rho}\zeta^1 + \frac{\partial L}{\partial \dot\phi}\zeta^2 + \frac{\partial L}{\partial \dot z}\zeta^3 - \Lambda \\
&= \frac{\partial L}{\partial \dot\phi} - \text{constant} && \left(\zeta^1 = \zeta^3 = 0,\ \zeta^2 = 1,\ \Lambda = \text{constant}\right) \\
&= \frac{m}{2}\left(0 + 2\dot\rho^2\dot\phi + 0\right) + 0 - \text{constant} \\
&= m\dot\rho^2\dot\phi - \text{constant}
\end{aligned}
$$

$\implies m\dot\rho^2\dot\phi =$ angular momentum about z-axis is conserved.

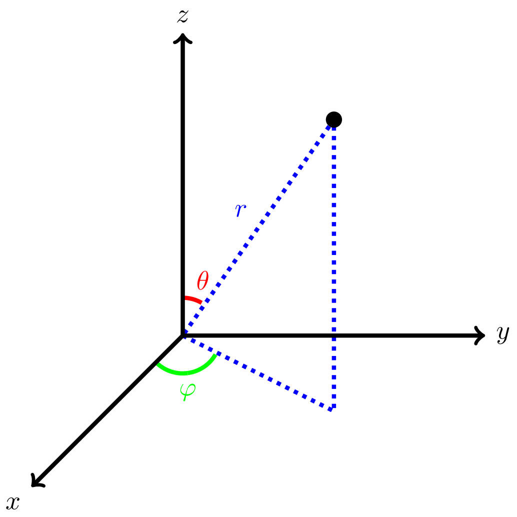
$$
\begin{aligned}
J_z &= \frac{\partial L}{\partial \dot\rho}\zeta^1 + \frac{\partial L}{\partial \dot\phi}\zeta^2 + \frac{\partial L}{\partial \dot z}\zeta^3 - \Lambda \\
&= \frac{\partial L}{\partial \dot z} - \text{constant} && \left(\zeta^1 = \zeta^2 = 0,\ \zeta^3 = 1,\ \Lambda = \text{constant}\right) \\
&= \frac{m}{2}\left(0 + 0 + 2\dot z\right) + 0 - \text{constant} \\
&= m\dot z - \text{constant}
\end{aligned}
$$

$\implies m\dot z =$ vertical momentum is conserved.

Problem 2

(a)

$$x = r\sin\theta\cos\varphi \qquad \dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$$
$$y = r\sin\theta\sin\varphi \qquad \dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$$
$$z = r\cos\theta \qquad \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$



$$L = \frac{m}{2}\left(\left(\dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi\right)^2 + \left(\dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi\right)^2 + \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2\right) - U(r)$$
$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - U(r)$$

(b)

L depends on r and θ, but not explicitly on ϕ, and so ϕ is the only cyclic coordinate.

$$\begin{aligned}
J_\phi &= \frac{\partial L}{\partial \dot{\rho}}\zeta^1 + \frac{\partial L}{\partial \dot{\phi}}\zeta^2 + \frac{\partial L}{\partial \dot{z}}\zeta^3 - \Lambda \\
&= \frac{\partial L}{\partial \dot{\phi}} - \text{constant} \qquad (\zeta^1 = \zeta^3 = 0,\ \zeta^2 = 1,\ \Lambda = \text{constant}) \\
&= \frac{m}{2}\left(0 + 0 + 2r^2\dot{\phi}\sin^2\theta\right) - 0 - \text{constant} \\
&= mr^2\dot{\phi}\sin^2\theta - \text{constant}
\end{aligned}$$

$\implies$ $m\left(r\sin\theta\right)^2\dot{\phi}$ = angular momentum about z-axis is conserved.

$\phi \to \tilde{\phi} = \phi + \epsilon$ is the corresponding continuous symmetry.

Problem 3

(a)

$$\begin{aligned} x_i \to x_i + \epsilon_{ij} x_j \implies r^2 &\to (x_i + \epsilon_{ij} x_j)^2 \\ &\to {x_i}^2 + 2\epsilon_{ij} x_i x_j + (\epsilon_{ij} x_j)^2 \\ &\to {x_i}^2 + (\epsilon_{ij} x_i x_j - \epsilon_{ji} x_j x_i) + \mathcal{O}\left({\epsilon_{ij}}^2\right) \\ &\to {x_i}^2 \end{aligned}$$

$$\begin{aligned} \dot{x}_i \to \dot{x}_i + \epsilon_{ij} \dot{x}_j \implies v^2 &\to (v_i + \epsilon_{ij} v_j)^2 \\ &\to {v_i}^2 + 2\epsilon_{ij} v_i v_j + (\epsilon_{ij} v_j)^2 \\ &\to {v_i}^2 + (\epsilon_{ij} v_i v_j - \epsilon_{ji} v_j v_i) + \mathcal{O}\left({\epsilon_{ij}}^2\right) \\ &\to {v_i}^2 \end{aligned}$$

L is a function of r^2 and v^2 which are both invariant under this infinitesimal rotation, and so L must also be invariant.

$$x_i \to \begin{cases} x_2 + \epsilon_{25} x_5, & i = 2 \\ x_5 - \epsilon_{25} x_2, & i = 5 \\ x_i \ \forall \ i \neq 2, 5 \end{cases}$$

$$v_i \to \begin{cases} v_2 + \epsilon_{25} v_5, & i = 2 \\ v_5 - \epsilon_{25} v_2, & i = 5 \\ v_i \ \forall \ i \neq 2, 5 \end{cases}$$

$$\begin{aligned} p_i &= \frac{\partial L}{\partial v_i} \\ &= \frac{\partial}{\partial v_i}\left(-mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}\right) - 0 \\ &= -mc^2\left(\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(0 - \frac{2v_i}{c^2}\right) \\ &= \frac{mv_i}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

$$\begin{aligned} p_i &\to \frac{m(v_i + \epsilon_{ij} v_j)}{\sqrt{1 - \frac{v^2}{c^2}}} \\ &\to p_i + \epsilon_{ij} p_j \end{aligned}$$

$$p_i \to \begin{cases} p_2 + \epsilon_{25} p_5, & i = 2 \\ p_5 - \epsilon_{25} p_2, & i = 5 \\ p_i \ \forall \ i \neq 2, 5 \end{cases}$$

(b)

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$p_4 = \frac{mv_4}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\begin{aligned} v^2 &= \frac{p^2}{m^2}\left(1 - \frac{v^2}{c^2}\right) \\ v^2\left(1 + \frac{p^2}{m^2c^2}\right) &= \frac{p^2}{m^2} \\ v^2 &= \frac{p^2}{m^2 + \frac{p^2}{c^2}} \\ \vec{v} &= \frac{\vec{p}}{\sqrt{m^2 + \frac{p^2}{c^2}}} \\ v_2 &= \frac{p_2}{\sqrt{m^2 + \frac{p^2}{c^2}}} \end{aligned}$$

(c)

$$J_{ij} = p_i x_j - p_j x_i, \;\; i,j = 1,2,\ldots,D$$

J_{ij} is a $D \times D$ anti-symmetric matrix, and so the diagonal contains only 0's, and half of the remaining terms are simply the same as the other half multiplied by -1. Thus there are $\frac{D^2-D}{2}$ independent terms in J_{ij}.

(d)

$$\begin{aligned} J_{ij} &= p_i x_j - p_j x_i, \;\; i,j = 1,2,3 \\ \vec{M} &= \vec{r} \times \vec{p} \\ J_{ij} &= -\varepsilon_{ijk} M_k \\ M_i &= -\frac{1}{2}\varepsilon_{ijk} J_{jk} \end{aligned}$$

(e)

$$\begin{aligned} E &= \frac{\partial L}{\partial v_i} v_i - L \\ E &= \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{\alpha}{r^2}e^{-\beta r^2} \\ &= \frac{p^2}{\sqrt{m^2 + \frac{p^2}{c^2}}} + mc^2\sqrt{1 - \frac{p^2}{c^2\left(m^2 + \frac{p^2}{c^2}\right)}} + \frac{\alpha}{r^2}e^{-\beta r^2} \\ E &= \frac{p^2}{\sqrt{m^2 + \frac{p^2}{c^2}}} + mc^2\sqrt{1 - \frac{p^2}{m^2c^2 + p^2}} + \frac{\alpha}{r^2}e^{-\beta r^2} \end{aligned}$$

Problem 4

(a)

To show that $O(2,1)$ is a group under standard matrix multiplication, we must show that it satisfies associativity, the identity property, the inverse property, and closure.

$$
\begin{aligned}
\text{For the following, } A, B, C \in O(2,1) & \\
\text{Associativity: } (AB)C &= A(BC) && \text{(standard matrix multiplication)} \\
\text{Identity: } I\eta I^T &= \eta I \\
&= \eta && \implies I \in O(2,1) \\
\text{Inverse: } A\eta A^T &= \eta \\
\eta &= A^{-1}\eta \left(A^T\right)^{-1} && \eta \text{ exists} \implies A^{-1} \text{ exists} \\
\text{Closure: } (AB)\eta(AB)^T &= AB\eta B^T A^T \\
&= A\eta A^T \\
&= \eta && \implies AB \in O(2,1)
\end{aligned}
$$

Thus $O(2,1)$ is a group under standard matrix multiplication.

(b)

$$
\begin{aligned}
A_{ij} &= \delta_{ij} + \epsilon_{ij} \\
A_{ik}\eta_{kn}A_{jn} &= \eta_{ij} \\
(\delta_{ik} + \epsilon_{ik})\eta_{kn}(\delta_{jn} + \epsilon_{jn}) &= \eta_{ij} \\
(\delta_{ik}\delta_{jn}\eta_{kn} + \delta_{ik}\epsilon_{jn}\eta_{kn} + \delta_{jn}\epsilon_{ik}\eta_{kn} + \epsilon_{ik}\epsilon_{jn}\eta_{kn}) &= \eta_{ij} \\
\eta_{ij} + \epsilon_{jn}\eta_{in} + \epsilon_{ik}\eta_{kj} + 0 &= \eta_{ij} \\
\epsilon_{jn}\eta_{in} + \epsilon_{ik}\eta_{kj} &= 0
\end{aligned}
$$

$$
\begin{aligned}
i=0,\ j=0 &\implies \epsilon_{00}\eta_{00} + \epsilon_{00}\eta_{00} = 0 && \implies \epsilon_{00} = 0 \\
i=0,\ j=1 &\implies \epsilon_{10}\eta_{00} + \epsilon_{01}\eta_{11} = 0 && \implies \epsilon_{01} = \epsilon_{10} \\
i=0,\ j=2 &\implies \epsilon_{20}\eta_{00} + \epsilon_{02}\eta_{22} = 0 && \implies \epsilon_{02} = \epsilon_{20} \\
i=1,\ j=1 &\implies \epsilon_{11}\eta_{11} + \epsilon_{11}\eta_{11} = 0 && \implies \epsilon_{11} = 0 \\
i=1,\ j=2 &\implies \epsilon_{21}\eta_{11} + \epsilon_{12}\eta_{22} = 0 && \implies \epsilon_{12} = -\epsilon_{21} \\
i=2,\ j=2 &\implies \epsilon_{22}\eta_{22} + \epsilon_{22}\eta_{22} = 0 && \implies \epsilon_{22} = 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
x_i \to A_{ij}x_j = x_i + \epsilon_{ij}x_j \implies {x_i}^2 &\to (x_i + \epsilon_{ij}x_j)^2 \\
&\to {x_i}^2 + 2\epsilon_{ij}x_i x_j + (\epsilon_{ij}x_j)^2 \\
&\to {x_i}^2 + (\epsilon_{ij}x_i x_j - \epsilon_{ji}x_j x_i) + \mathcal{O}\left({\epsilon_{ij}}^2\right) \\
&\to {x_i}^2 \qquad \text{fixed } i = 0,1,2
\end{aligned}
$$

$$
\begin{aligned}
v_i \to A_{ij}v_j = v_i + \epsilon_{ij}v_j \implies {v_i}^2 &\to (v_i + \epsilon_{ij}v_j)^2 \\
&\to {v_i}^2 + 2\epsilon_{ij}v_i v_j + (\epsilon_{ij}v_j)^2 \\
&\to {v_i}^2 + (\epsilon_{ij}v_i v_j - \epsilon_{ji}v_j v_i) + \mathcal{O}\left({\epsilon_{ij}}^2\right) \\
&\to {v_i}^2 \qquad \text{fixed } i = 0,1,2
\end{aligned}
$$

L is a function of ${x_i}^2$ and ${v_i}^2$, fixed $i = 0,1,2$ which are both invariant under this infinitesimal rotation, and so L must also be invariant.

(d)

$x_i \to A_{ij}x_j = x_i + \epsilon_{ij}x_j$ is a continuous symmetry, as it does not change L.
L is invariant under rotation $\implies$ angular momentum $\vec{M} = m\vec{r} \times \vec{v}$ is conserved.
L does not explicitly depend on t $\implies$ energy $E = \frac{m}{2}\left(-{v_0}^2 + {v_1}^2 + {v_2}^2\right)$ is conserved.

(e)

$$
\begin{aligned}
-{x_0}^2 + {x_1}^2 + {x_2}^2 + a^2 &= 0 \\
-r^2\cosh^2\zeta + r^2\cos^2\phi\sinh^2\zeta + r^2\sin^2\phi\sinh^2\zeta &= -a^2 \\
-r^2\cosh^2\zeta + r^2\sinh^2\zeta &= -a^2 \\
r &= a
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= a\cosh\zeta & v_0 &= a\dot\zeta\sinh\zeta \\
x_1 &= a\cos\phi\sinh\zeta & v_1 &= -a\dot\phi\sin\phi\sinh\zeta + a\dot\zeta\cos\phi\cosh\zeta \\
x_2 &= a\sin\phi\sinh\zeta & v_2 &= a\dot\phi\cos\phi\sinh\zeta + a\dot\zeta\sin\phi\cosh\zeta
\end{aligned}
$$

$$
\begin{aligned}
L = \frac{m}{2}\Big(&-a^2\dot\zeta^2\sinh^2\zeta + a^2\dot\phi^2\sin^2\phi\sinh^2\zeta + a^2\dot\zeta^2\cos^2\phi\cosh^2\zeta - 2a^2\dot\phi\dot\zeta\sin\phi\cos\phi\sinh\zeta\cosh\zeta \\
&+ a^2\dot\phi^2\cos^2\phi\sinh^2\zeta + a^2\dot\zeta^2\sin^2\phi\cosh^2\zeta + 2a^2\dot\phi\dot\zeta\sin\phi\cos\phi\sinh\zeta\cosh\zeta\Big) \\
&+ \lambda\left(-r^2\cosh^2\zeta + r^2\cos^2\phi\sinh^2\zeta + r^2\sin^2\phi\sinh^2\zeta + a^2\right) \\
= \frac{m}{2}&\left(a^2\dot\zeta^2\left(\cosh^2\zeta\left(\cos^2\phi + \sin^2\phi\right) - \sinh^2\zeta\right) + a^2\dot\phi^2\sinh^2\zeta\left(\cos^2\phi + \sin^2\phi\right)\right) + \lambda\left(a^2 - r^2\right) \\
L = \frac{m}{2}&\left(a^2\dot\zeta^2 + a^2\dot\phi^2\sinh^2\zeta\right) + \lambda\left(a^2 - r^2\right)
\end{aligned}
$$

$$
M_i = m\varepsilon_{ijk}r_jv_k
$$

$$
\begin{aligned}
M_0 &= m(x_1v_2 - x_2v_1) \\
&= m\left(a^2\dot\phi\cos^2\phi\sinh^2\zeta + a^2\dot\zeta\cos\phi\sin\phi\cosh\zeta\sinh\zeta + a^2\dot\phi\sin^2\phi\sinh^2\zeta - a^2\dot\zeta\cos\phi\sin\phi\cosh\zeta\sinh\zeta\right) \\
&= ma^2\dot\phi\sinh^2\zeta
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= m(x_2v_0 - x_0v_2) \\
&= m\left(a^2\dot\zeta\sin\phi\sinh^2\zeta - a^2\dot\phi\cos\phi\cosh\zeta\sinh\zeta - a^2\dot\zeta\sin\phi\cosh^2\zeta\right) \\
&= ma^2\left(-\dot\zeta\sin\phi - \dot\phi\cos\phi\cosh\zeta\sinh\zeta\right)
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= m(x_0v_1 - x_1v_0) \\
&= m\left(-a^2\dot\phi\sin\phi\cosh\zeta\sinh\zeta + a^2\dot\zeta\cos\phi\cosh^2\zeta - a^2\dot\zeta\cos\phi\sinh^2\zeta\right) \\
&= ma^2\left(\dot\zeta\cos\phi - \dot\phi\sin\phi\cosh\zeta\sinh\zeta\right)
\end{aligned}
$$

$$
\begin{aligned}
\implies \vec{M} &= ma^2\begin{pmatrix} \dot\phi\sinh^2\zeta \\ -\dot\zeta\sin\phi - \dot\phi\cos\phi\cosh\zeta\sinh\zeta \\ \dot\zeta\cos\phi - \dot\phi\sin\phi\cosh\zeta\sinh\zeta \end{pmatrix} \\
E &= \frac{m}{2}\left(a^2\dot\zeta^2 + a^2\dot\phi^2\sinh^2\zeta\right)
\end{aligned}
$$

(f)

(g)

$$
\begin{aligned}
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\
&= \frac{m}{2}\left(0 + 2a^2\dot{\phi}\sinh^2\zeta\right) \\
&= ma^2\dot{\phi}\sinh^2\zeta \\
p_\zeta &= \frac{\partial L}{\partial \dot{\zeta}} \\
&= \frac{m}{2}\left(2a^2\dot{\zeta}\right) \\
&= ma^2\dot{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
\vec{M} &= \begin{pmatrix} p_\phi \\ -p_\zeta \sin\phi - p_\phi \cos\phi \coth\zeta \\ p_\zeta \cos\phi - p_\phi \sin\phi \coth\zeta \end{pmatrix} \\
E &= \frac{1}{2ma^2}\left({p_\zeta}^2 + \frac{{p_\phi}^2}{\sinh^2\zeta}\right)
\end{aligned}
$$