MAU23401: Advanced Classical Mechanics I
Homework 2 due 16/10/20

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SF Theoretical Physics

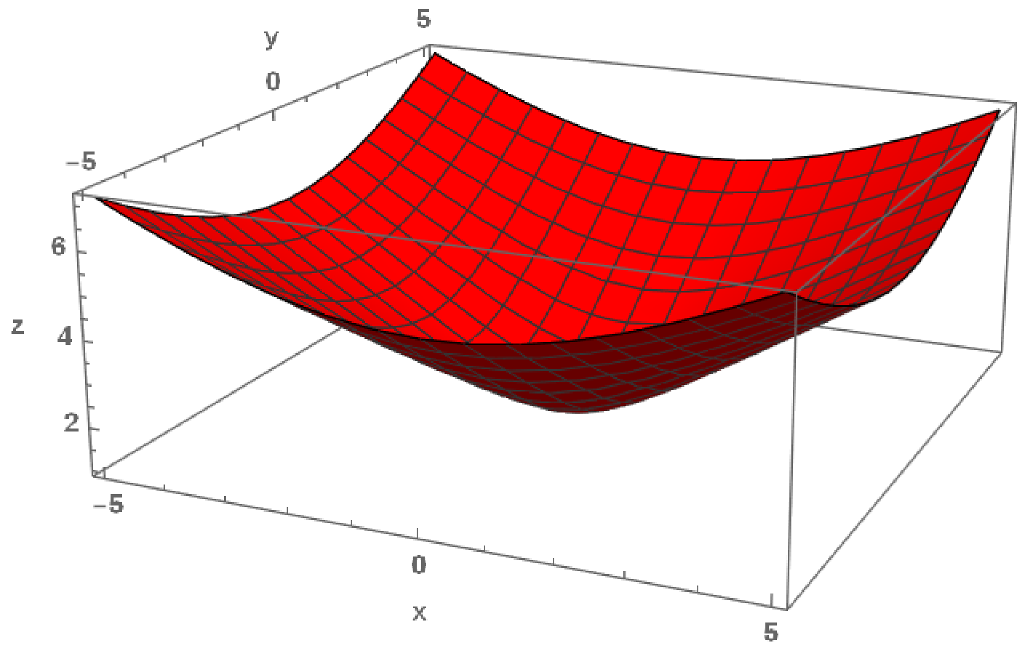
I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

(a)

This surface is the top half of a two-sheeted hyperboloid.

```
k := 1
c1 := 1
f1[x_, y_] := k*Sqrt[x^2 + y^2 + c1^2]
Plot3D[f1[x, y], {x, -5, 5}, {y, -5, 5}, PlotStyle → {Red},
 AxesLabel → {"x", "y", "z"}, LabelStyle → {Bold}]
```



(b)

$$\begin{aligned} L_A &= T_A - U_A \\ &= \frac{1}{2} m v^2 - mgz \\ &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \end{aligned}$$

$$\begin{aligned} x &= r\cos\phi & \dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ y &= r\sin\phi & \dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ z &= k\sqrt{x^2 + y^2 + c^2} \\ &= k\sqrt{r^2\cos^2\phi + r^2\sin^2\phi + c^2} \\ &= k\sqrt{r^2 + c^2} & \dot{z} &= \frac{kr\dot{r}}{\sqrt{r^2 + c^2}} \end{aligned}$$

$$\begin{aligned} L = \frac{m}{2} & \left(\dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + \frac{k^2r^2\dot{r}^2}{r^2 + c^2}\right) \\ & - mgk\sqrt{r^2 + c^2} \end{aligned}$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \frac{k^2r^2\dot{r}^2}{r^2 + c^2}\right) - mgk\sqrt{r^2 + c^2}$$

(c)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= \frac{m}{2}\left(2\dot{r} + 0 + \frac{2k^2r^2\dot{r}}{r^2+c^2}\right) - 0 \\
&= m\left(\dot{r} + \frac{k^2r^2\dot{r}}{r^2+c^2}\right) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\left(\ddot{r} + k^2\frac{d}{dt}\left(\frac{r^2\dot{r}}{r^2+c^2}\right)\right) \\
&= m\left(\ddot{r} + k^2\left(\frac{2r\dot{r}}{r^2+c^2} - \frac{2r^3\dot{r}}{(r^2+c^2)^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial r} &= \frac{m}{2}\left(0 + 2r\dot{\phi}^2 + k^2\dot{r}^2\left(\frac{2r}{r^2+c^2} - \frac{2r^3}{(r^2+c^2)^2}\right)\right) - \frac{mgkr}{\sqrt{r^2+c^2}} \\
&= m\left(r\dot{\phi}^2 + \frac{k^2r\dot{r}^2}{r^2+c^2} - \frac{k^2r^3\dot{r}^2}{(r^2+c^2)^2} - \frac{gkr}{\sqrt{r^2+c^2}}\right)
\end{aligned}$$

$$\begin{aligned}
\implies \ddot{r} + \frac{k^2r^2\ddot{r}}{r^2+c^2} + \frac{2k^2r\dot{r}^2}{r^2+c^2} - \frac{2k^2r^2\dot{r}^2}{(r^2+c^2)^2} &= r\dot{\phi}^2 + \frac{k^2r\dot{r}^2}{r^2+c^2} - \frac{k^2r^3\dot{r}^2}{(r^2+c^2)^2} - \frac{gkr}{\sqrt{r^2+c^2}} \\
\ddot{r} + \frac{k^2r^2\ddot{r}}{r^2+c^2} + \frac{k^2r\dot{r}^2}{r^2+c^2} + \frac{gkr}{\sqrt{r^2+c^2}} &= r\dot{\phi}^2 + \frac{k^2r^2\dot{r}^2}{(r^2+c^2)^2}
\end{aligned}$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\frac{\partial L}{\partial \phi} = 0$$

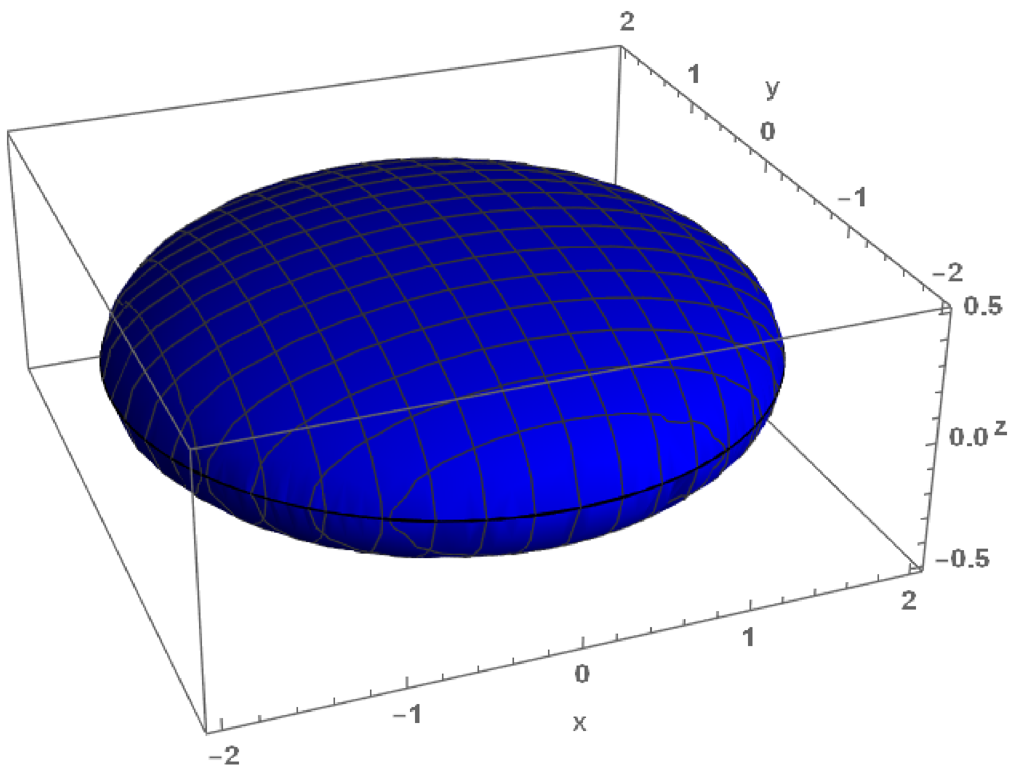
$$\begin{aligned}
\implies \frac{d}{dt}\left(mr^2\dot{\phi}\right) &= 0 \\
\implies r^2\dot{\phi} &= \text{constant}
\end{aligned}$$

Problem 2

(a)

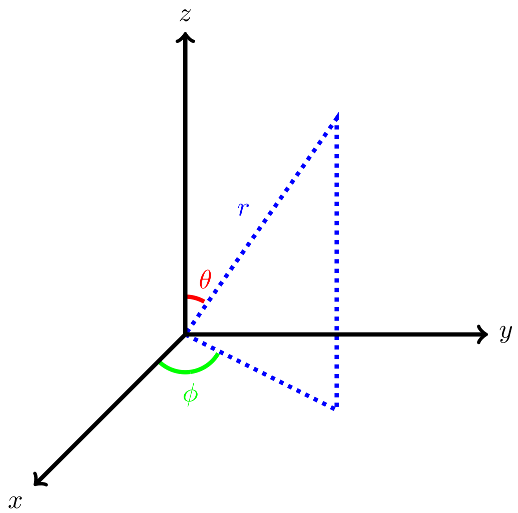
This surface is a spheroid.

```
a := 2
c2 := 1 / 2
f21[x_, y_] := c2 * Sqrt[1 - (x^2 + y^2) / a^2]
f22[x_, y_] := c2 * -Sqrt[1 - (x^2 + y^2) / a^2]
Plot3D[{f21[x, y], f22[x, y]}, {x, -2, 2}, {y, -2, 2},
 PlotStyle → {Blue, Blue}, AxesLabel → {"x", "y", "z"},
 LabelStyle → {Bold}]
```



(b)

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$



(c)

$$\begin{aligned}
x &= a\rho \sin\theta\cos\phi & \dot{x} &= a\rho\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)\\
y &= a\rho \sin\theta\sin\phi & \dot{y} &= a\rho\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)\\
z &= c\rho\cos\theta & \dot{z} &= -c\rho\dot{\theta}\sin\theta
\end{aligned}$$

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} &= 1\\
\rho^2\sin^2\theta\cos^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta &= 1\\
\rho^2\sin^2\theta + \rho^2\cos^2\theta &= 1\\
\implies \rho &= 1
\end{aligned}$$

$$\begin{aligned}
x &= a\sin\theta\cos\phi & \dot{x} &= a\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)\\
y &= a\sin\theta\sin\phi & \dot{y} &= a\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)\\
z &= c\cos\theta & \dot{z} &= -c\dot{\theta}\sin\theta
\end{aligned}$$

$$\begin{aligned}
L &= T - U\\
&= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz\\
&= \frac{m}{2}\Big(a^2\Big(\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi - 2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi + \dot{\theta}^2\cos^2\theta\sin^2\theta\\
&\quad + \dot{\phi}^2\sin^2\theta\cos^2\phi + 2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi\Big) + c^2\dot{\theta}^2\sin^2\theta\Big) - mgc\cos\theta
\end{aligned}$$

$$L = \frac{m}{2}\left(a^2\left(\dot{\theta}^2\cos^2\theta + \dot{\phi}^2\sin^2\theta\right) + c^2\dot{\theta}^2\sin^2\theta\right) - mgc\cos\theta$$

(d)

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{m}{2}\left(a^2\left(2\dot{\theta}\cos^2\theta + 0\right) + 2c^2\dot{\theta}\sin^2\theta\right) - 0 \\
&= m\dot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= m\left(\ddot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) + \dot{\theta}\left(a^2\left(2\cos\theta\right)\left(-\dot{\theta}\sin\theta\right) + c^2\left(2\sin\theta\right)\left(\dot{\theta}\cos\theta\right)\right)\right) \\
&= m\left(\ddot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) + 2\dot{\theta}^2\cos\theta\sin\theta\left(c^2 - a^2\right)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= \frac{m}{2}\left(a^2\left(-2\dot{\theta}^2\cos\theta\sin\theta + 2\dot{\phi}^2\cos\theta\sin\theta\right) + 2c^2\dot{\theta}^2\cos\theta\sin\theta\right) + mgc\sin\theta \\
&= m\left(a^2\cos\theta\sin\theta\left(\dot{\phi}^2 - \dot{\theta}^2\right) + c^2\dot{\theta}^2\cos\theta\sin\theta + gc\sin\theta\right)
\end{aligned}$$

$$\begin{aligned}
\ddot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) + 2\dot{\theta}^2\cos\theta\sin\theta\left(c^2 - a^2\right) &= a^2\cos\theta\sin\theta\left(\dot{\phi}^2 - \dot{\theta}^2\right) + c^2\dot{\theta}^2\cos\theta\sin\theta + gc\sin\theta \\
\ddot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) + c^2\dot{\theta}^2\cos\theta\sin\theta &= a^2\dot{\phi}^2\cos\theta\sin\theta + a^2\dot{\theta}^2\cos\theta\sin\theta + gc\sin\theta \\
\ddot{\theta}\left(a^2\cos^2\theta + c^2\sin^2\theta\right) + c^2\dot{\theta}^2\cos\theta\sin\theta &= a^2\cos\theta\sin\theta\left(\dot{\theta}^2 + \dot{\phi}^2\right) + gc\sin\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\phi}} &= \frac{m}{2}\left(a^2\left(0 + 2\dot{\phi}\sin^2\theta\right) + 0\right) - 0 \\
&= ma^2\dot{\phi}\sin^2\theta
\end{aligned}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\begin{aligned}
\implies \frac{d}{dt}\left(ma^2\dot{\phi}\sin^2\theta\right) &= 0 \\
\implies \dot{\phi}\sin^2\theta &= \text{constant}
\end{aligned}$$

Problem 3

(a)

$$\begin{aligned}
L_A &= T_A - U_A \\
&= \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 - m_1gz_1 - m_2gz_2 \\
&= \frac{m_1}{2}\left({\dot{x}_1}^2 + {\dot{z}_1}^2\right) + \frac{m_2}{2}\left({\dot{x}_2}^2 + {\dot{z}_2}^2\right) - m_1gz_1 - m_2gz_2 \\
&= \frac{m_1\dot{q}^2}{2}\left(f'^2 + h'^2\right) + \frac{m_2}{2}\left({\dot{x}_2}^2 + {\dot{z}_2}^2\right) - m_1gz_1 - m_2gz_2
\end{aligned}$$

$$\begin{aligned}
l &\equiv \text{distance between masses} \\
&= \sqrt{({x_2}^2 - {x_1}^2) + ({z_2}^2 - {z_1}^2)}
\end{aligned}$$

$$\begin{aligned}
x_2 &= f + l\sin\phi & \dot{x}_2 &= \dot{q}f' + l\dot{\phi}\cos\phi \\
z_2 &= h - l\cos\phi & \dot{z}_2 &= \dot{q}h' + l\dot{\phi}\sin\phi
\end{aligned}$$

$$\begin{aligned}
L = &\frac{m_1\dot{q}^2}{2}\left(f'^2 + h'^2\right) + \frac{m_2}{2}\left(\dot{q}^2f'^2 + l^2\dot{\phi}^2\cos^2\phi + 2f'\dot{q}l\dot{\phi}\cos\phi + \dot{q}^2h'^2 + l^2\dot{\phi}^2\sin^2\phi + 2h'\dot{q}l\dot{\phi}\sin\phi\right) \\
&- m_1gh - m_2g(h - l\cos\phi)
\end{aligned}$$

$$L = \frac{m_1\dot{q}^2}{2}\left(f'^2 + h'^2\right) + \frac{m_2}{2}\left(\dot{q}^2\left(f'^2 + h'^2\right) + l^2\dot{\phi}^2 + 2\dot{q}l\dot{\phi}\left(f'\cos\phi + h'\sin\phi\right)\right) - m_1gh - m_2g(h - l\cos\phi)$$

(b)

$$q \to s \implies \begin{cases} \dot{q}^2\left(f'^2(q)+h'^2(q)\right) = \dot{x}^2+\dot{z}^2 \to \dot{s}^2 \\ \dot{q}f'(q) \to \dot{s}f'(s) \\ \dot{q}h'(q) \to \dot{s}h'(s) \end{cases}$$

$$\implies L = \frac{m_1\dot{s}^2}{2} + \frac{m_2}{2}\left(\dot{s}^2 + l^2\dot{\phi}^2 + 2\dot{s}l\dot{\phi}\left(f'\cos\phi + h'\sin\phi\right)\right) - m_1gh - m_2g(h - l\cos\phi)$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{s}} &= m_1\dot{s} + \frac{m_2}{2}\left(2\dot{s} + 0 + 2l\dot{\phi}\left(f'\cos\phi + h'\sin\phi\right)\right) - 0 - 0 \\ &= m_1\dot{s} + m_2\left(\dot{s} + l\dot{\phi}\left(f'\cos\phi + h'\sin\phi\right)\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} &= m_1\ddot{s} + m_2\left(\ddot{s} + l\ddot{\phi}\left(f'\cos\phi + h'\sin\phi\right) + l\dot{\phi}\left(\dot{s}f''\cos\phi - \dot{\phi}f'\sin\phi + \dot{s}h''\sin\phi + \dot{\phi}h'\cos\phi\right)\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial s} &= 0 + \frac{m_2}{2}\left(0 + 0 + 2\dot{s}l\dot{\phi}\left(f''\cos\phi + h''\sin\phi\right)\right) - m_1gh' - m_2gh' \\ &= m_2\dot{s}l\dot{\phi}\left(f''\cos\phi + h''\sin\phi\right) - gh'(m_1+m_2) \end{aligned}$$

$$\begin{aligned} m_1\ddot{s} + m_2\left(\ddot{s} + l\ddot{\phi}\left(f'\cos\phi + h'\sin\phi\right) + l\dot{\phi}\left(\dot{s}f''\cos\phi - \dot{\phi}f'\sin\phi + \dot{s}h''\sin\phi + \dot{\phi}h'\cos\phi\right)\right) \\ = m_2\dot{s}l\dot{\phi}\left(f''\cos\phi + h''\sin\phi\right) - gh'(m_1+m_2) \end{aligned}$$

$$m_1\ddot{s} + m_2\left(\ddot{s} + l\ddot{\phi}\left(f'\cos\phi + h'\sin\phi\right) + l\dot{\phi}^2\left(h'\cos\phi - f'\sin\phi\right)\right) + gh'(m_1+m_2) = 0$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= 0 + \frac{m_2}{2}\left(0 + 2l^2\dot{\phi} + 2\dot{s}l\left(f'\cos\phi + h'\sin\phi\right)\right) - 0 - 0 \\ &= m_2l\left(l\dot{\phi} + \dot{s}\left(f'\cos\phi + h'\sin\phi\right)\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= m_2l\left(l\ddot{\phi} + \ddot{s}\left(f'\cos\phi + h'\sin\phi\right) + \dot{s}\left(\dot{s}f''\cos\phi - \dot{\phi}f'\sin\phi + \dot{s}h''\sin\phi + \dot{\phi}h'\cos\phi\right)\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \phi} &= 0 + \frac{m_2}{2}\left(0 + 0 + 2\dot{s}l\dot{\phi}\left(-f'\sin\phi + h'\cos\phi\right)\right) - 0 - m_2g(0 + l\sin\phi) \\ &= m_2l\left(\dot{s}\dot{\phi}\left(-f'\sin\phi + h'\cos\phi\right) - g\sin\phi\right) \end{aligned}$$

$$\begin{aligned} l\ddot{\phi} + \ddot{s}\left(f'\cos\phi + h'\sin\phi\right) + \dot{s}\left(\dot{s}f''\cos\phi - \dot{\phi}f'\sin\phi + \dot{s}h''\sin\phi + \dot{\phi}h'\cos\phi\right) = \dot{s}\dot{\phi}\left(-f'\sin\phi + h'\cos\phi\right) \\ - g\sin\phi \end{aligned}$$

$$l\ddot{\phi} + \ddot{s}\left(f'\cos\phi + h'\sin\phi\right) + \dot{s}^2\left(f''\cos\phi + h''\sin\phi\right) + g\sin\phi = 0$$

(c)

$$x = a\cos\beta \qquad\qquad z = c\sin\beta$$

$$\begin{aligned}\frac{x^2}{a^2}+\frac{z^2}{c^2} &= \cos^2\beta+\sin^2\beta \\ &= 1\end{aligned}$$

$$q \to \beta \implies \begin{cases} x = f(q) \to x = f(\beta) = a\cos\beta \\ z = h(q) \to z = h(\beta) = c\sin\beta \\ \dot{q} \to \dot{\beta} \\ f'(q) \to f'(\beta) = -a\sin\beta \\ h'(q) \to h'(\beta) = c\cos\beta \end{cases}$$

$$\begin{aligned} L = {} & \frac{m_1\dot{\beta}^2}{2}\left(a^2\sin^2\beta + c^2\cos^2\beta\right) + \frac{m_2}{2}\Big(\dot{\beta}^2\left(a^2\sin^2\beta + c^2\cos^2\beta\right) + l^2\dot{\phi}^2 \\ & +2\dot{\beta}l\dot{\phi}\left(-a\sin\beta\cos\phi + c\cos\beta\sin\phi\right)\Big) - m_1gc\sin\beta - m_2g\left(c\sin\beta - l\cos\phi\right) \end{aligned}$$

$$\begin{aligned} s(\beta) &= \int_{\beta_1}^{\beta_2}\sqrt{\left(\frac{dx}{d\beta}\right)^2+\left(\frac{dz}{d\beta}\right)^2}\,d\beta \\ &= \int_{\beta_1}^{\beta_2}\sqrt{a^2\sin^2\beta + c^2\cos^2\beta}\,d\beta \end{aligned}$$

```
Integrate[Sqrt[(a*Sin[β])^2 + (c*Cos[β])^2], {β, β1, β2}]
```

$$-\frac{\sqrt{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta1]}\ \mathrm{EllipticE}\left[\beta1,\,1-\frac{a^2}{c^2}\right]}{\sqrt{\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta1]}{c^2}}} + \frac{\sqrt{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta2]}\ \mathrm{EllipticE}\left[\beta2,\,1-\frac{a^2}{c^2}\right]}{\sqrt{\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta2]}{c^2}}} \quad \text{if } \textit{condition}$$

$(a^2-c^2)\,\mathrm{Cos}[2\,\beta2] \neq a^2+c^2$ &&

$\left(\mathrm{Re}\left[\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta1]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])}\right] > 0 \;\|\; 1+\mathrm{Re}\left[\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta1]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])}\right] < 0 \;\|\; \frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta1]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])} \notin \mathbb{R}\right)$ &&

$\left(\mathrm{Re}\left[\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta2]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])}\right] > 1 \;\|\; \mathrm{Re}\left[\frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta2]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])}\right] < 0 \;\|\; \frac{a^2+c^2+(-a^2+c^2)\,\mathrm{Cos}[2\,\beta2]}{(a^2-c^2)\,(\mathrm{Cos}[2\,\beta1]-\mathrm{Cos}[2\,\beta2])} \notin \mathbb{R}\right)$ &&

$((2\pi c_1 == \mathrm{Re}[\beta1]$ && $2\pi c_1 == \mathrm{Re}[\beta2]$ &&

$((\mathrm{Im}[\beta1] < 0$ && $\mathrm{Abs}[\mathrm{Im}[\beta1]] + \mathrm{Im}[\beta1] \le 2\,\mathrm{Im}[\beta2]) \;\|\; (\mathrm{Im}[\beta1] > 0$ && $\mathrm{Abs}[\mathrm{Im}[\beta1]] + 2\,\mathrm{Im}[\beta2] \le \mathrm{Im}[\beta1]) \;\|\; \beta1 \in \mathbb{R})) \;\|$

$(2\pi c_1\,\mathrm{Im}[\beta1] + \mathrm{Im}[\beta2]\,\mathrm{Re}[\beta1] == 2\pi c_1\,\mathrm{Im}[\beta2] + \mathrm{Im}[\beta1]\,\mathrm{Re}[\beta2]$ &&

$((2\pi c_1 < \mathrm{Re}[\beta1]$ && $2\pi c_1 \ge \mathrm{Re}[\beta2]) \;\|\; (2\pi c_1 > \mathrm{Re}[\beta1]$ && $2\pi c_1 \le \mathrm{Re}[\beta2]))))$ && $c_1 \in \mathbb{Z}$