MAU23401: Advanced Classical Mechanics I
Homework 1 due 09/10/20

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I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

(a)

$$\begin{aligned} U(r) &= \frac{k^2}{4g} - \frac{k}{2}r^2 + \frac{g}{4}r^4 \\ \frac{dU}{dr} &= 0 - kr + gr^3 \\ &= 0 \\ \implies r(gr^2 - k) &= 0 \end{aligned}$$

$$r = 0 \qquad\qquad r = \pm\sqrt{\frac{k}{g}}$$

$$U(0) = \frac{k^2}{4g} \qquad\qquad \begin{aligned} U\left(\pm\sqrt{\frac{k}{g}}\right) &= \frac{k^2}{4g} - \frac{k^2}{2g} + \frac{k^2}{4g} \\ &= 0 \end{aligned}$$

$$\frac{d^2U}{dr^2} = -k + 3gr^2$$

$$\begin{aligned} \left.\frac{d^2U}{dr^2}\right|_{r=0} &= -k \\ &< 0 \end{aligned} \qquad\qquad \begin{aligned} \left.\frac{d^2U}{dr^2}\right|_{r=\pm\sqrt{\frac{k}{g}}} &= -k + 3k = 2k \\ &> 0 \end{aligned}$$

$$\text{maximum potential} = \frac{k^2}{4g},\ r = 0$$

$$\text{minimum potential} = 0,\ r = \pm\sqrt{\frac{k}{g}}$$

```
U1[x_] := (1 / 4) - (1 / 2) * x^2 + (1 / 4) * x^4
U2[x_, y_] := (1 / 4) - (1 / 2) * (x^2 + y^2) + (1 / 4) * (x^2 + y^2) ^2
Plot[U1[x], {x, -1.5, 1.5}, PlotStyle → {Blue, Thick},
 AxesLabel → {"x", "U(x)"}, LabelStyle → {Bold}]
Plot3D[U2[x1, x2], {x1, -1.5, 1.5}, {x2, -1.25, 1.25},
 PlotStyle → {Red}, AxesLabel → {"x1", "x2", "U(r)"},
 LabelStyle → {Bold}]
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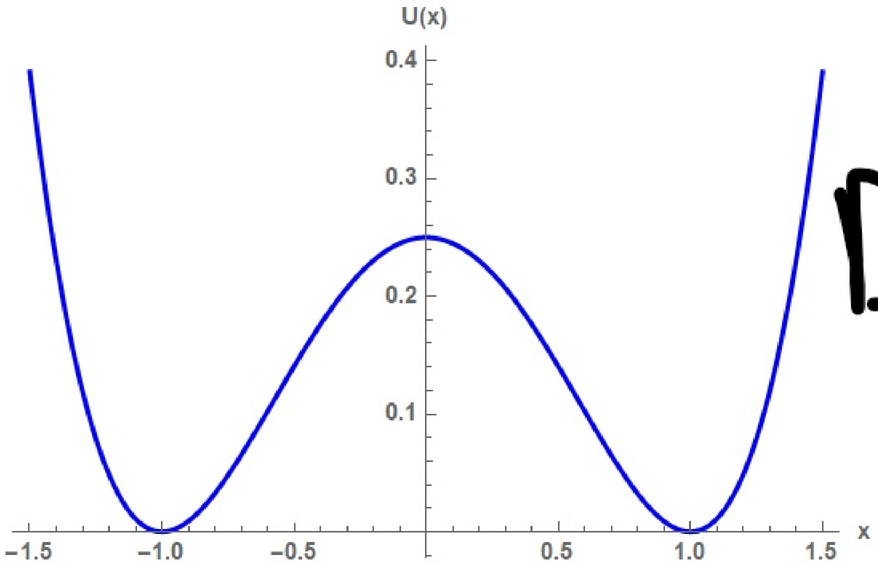
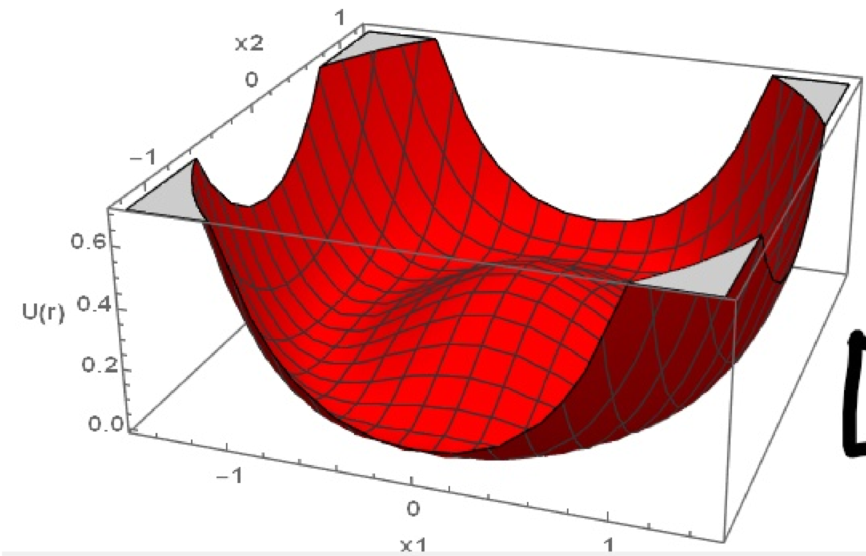


D=1



D=2

(b)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$$

$$\begin{aligned}
L &= \frac{1}{2}m{v_i}^2 - \frac{k^2}{4g} + \frac{k}{2}r^2 - \frac{g}{4}r^4 \\
&= \frac{1}{2}m{\dot{x}_j}^2 - \frac{k^2}{4g} + \frac{k}{2}{x_j}^2 - \frac{g}{4}\left({x_j}^2\right)^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}_i} &= \frac{1}{2}m(2\delta_{ij}\dot{x}_j) - 0 + 0 - 0 \\
&= m\dot{x}_i \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} &= m\ddot{x}_i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= 0 - 0 + \frac{k}{2}(2\delta_{ij}x_j) - \frac{g}{4}\left(2{x_j}^2\right)\left(2\delta_{ij}x_j\right) \\
&= kx_i - g{x_j}^2 x_i \\
&= x_i\left(k - gr^2\right)
\end{aligned}$$

$$\begin{aligned}
\implies m\ddot{x}_i &= x_i\left(k - gr^2\right) \\
\ddot{x}_i &= \frac{k - gr^2}{m}x_i
\end{aligned}$$

(c)

L is a function of r^2 and ${v_i}^2$. If we can show that r^2 and ${v_i}^2$ are each invariant under the change of coordinates, then we can say that L must be also.

$$\begin{aligned}
x_i \to \mathcal{O}_{ij}x_j \implies r^2 = {x_i}^2 &\to (\mathcal{O}_{ij}x_j)(\mathcal{O}_{ik}x_k) \\
&\to \mathcal{O}_{ij}\mathcal{O}_{ik}x_jx_k \\
&\to \delta_{jk}x_jx_k \\
&\to x_jx_j \\
&\to {x_j}^2 \\
r^2 &\to r^2 \implies r^2 \text{ is invariant under this change}
\end{aligned}$$

$$\begin{aligned}
\dot{x}_i \to \mathcal{O}_{ij}\dot{x}_j \implies {v_i}^2 = {\dot{x}_i}^2 &\to (\mathcal{O}_{ij}\dot{x}_j)(\mathcal{O}_{ik}\dot{x}_k) \\
&\to \mathcal{O}_{ij}\mathcal{O}_{ik}\dot{x}_j\dot{x}_k \\
&\to \delta_{jk}\dot{x}_j\dot{x}_k \\
&\to \dot{x}_j\dot{x}_j \\
&\to {\dot{x}_j}^2 \\
{v_i}^2 &\to {v_i}^2 \implies {v_i}^2 \text{ is invariant under this change}
\end{aligned}$$

Therefore we can say that L is invariant under this change of coordinates.

Problem 2

(a)

$$m_{ij}\dot{q}^i\dot{q}^j = m_{ji}\dot{q}^j\dot{q}^i \qquad\qquad k_{ij}e^{q^i+q^j} = k_{ji}e^{q^j+q^i}$$
$$= m_{ji}\dot{q}^i\dot{q}^j \qquad\qquad = k_{ji}e^{q^i+q^j}$$

We can assume that $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$ without a loss of generality as this does not change the expressions above and thus does not change L.

$$\begin{aligned}
L_1 &= \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j \\
m_{ij} \text{ is symmetric } &\implies m_{ij} \text{ can be diagonalised} \\
m_{ij} &\to \mathrm{diag}(\lambda_1, \ldots, \lambda_s),\ \lambda_i \equiv i^{\text{th}} \text{ eigenvalue of } m_{ij} \\
\implies L &\to \sum_{i=1}^{s} \frac{\lambda_i}{2}\dot{Q}^i\dot{Q}^i \\
&> 0 \\
\frac{\dot{Q}^i\dot{Q}^i}{2} &> 0\ \forall\ i = 1, \ldots, s \\
\implies \lambda_i &> 0\ \forall\ i = 1, \ldots, s
\end{aligned}$$

$\therefore$ m_{ij} must be positive definite.

(b)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^n} = \frac{\partial L}{\partial q^n}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}^n} &= \frac{\partial}{\partial \dot{q}^n}\left(\frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j + b_{ij}\dot{q}^i q^j - \frac{1}{2}k_{ij}e^{q^i+q^j}\right) \\
&= \frac{1}{2}m_{ij}\delta_{in}\dot{q}^j + \frac{1}{2}m_{ij}\delta_{jn}\dot{q}^i + b_{ij}\delta_{in}q^j - 0 \\
&= \frac{1}{2}m_{nj}\dot{q}^j + \frac{1}{2}m_{in}\dot{q}^i + b_{nj}q^j \\
&= \frac{1}{2}m_{in}\dot{q}^i + \frac{1}{2}m_{in}\dot{q}^i + b_{ni}q^i \\
&= m_{in}\dot{q}^i + b_{ni}q^i \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^n} &= m_{in}\ddot{q}^i + b_{ni}\dot{q}^i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial q^n} &= \frac{\partial}{\partial q^n}\left(\frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j + b_{ij}\dot{q}^i q^j - \frac{1}{2}k_{ij}e^{q^i+q^j}\right) \\
&= 0 + b_{ij}\dot{q}^i\delta_{jn} - \frac{1}{2}k_{ij}\left(\frac{\partial}{\partial q^n}e^{q^i}\right)e^{q^j} - \frac{1}{2}k_{ij}e^{q^i}\left(\frac{\partial}{\partial q^n}e^{q^j}\right) \\
&= b_{in}\dot{q}^i - \frac{1}{2}k_{nj}e^{q^n}e^{q^j} - \frac{1}{2}k_{in}e^{q^i}e^{q^n} \qquad \text{(fixed } n\text{)} \\
&= b_{in}\dot{q}^i - \frac{1}{2}k_{in}e^{q^i}e^{q^n} - \frac{1}{2}k_{in}e^{q^i}e^{q^n} \\
&= b_{in}\dot{q}^i - k_{in}e^{q^i+q^n}
\end{aligned}$$

$$\implies m_{in}\ddot{q}^i + b_{ni}\dot{q}^i = b_{in}\dot{q}^i - k_{in}e^{q^i+q^n}$$

$$m_{in}\ddot{q}^i + (b_{ni} - b_{in})\,\dot{q}^i + k_{in}e^{q^i+q^n} = 0 \qquad \text{(fixed } n\text{)}$$

(c)

$$\begin{aligned}
b_{ij} &= \text{symmetric part of } b_{ij} + \text{anti-symmetric part of } b_{ij} \\
&= \frac{1}{2}(b_ij + b_ji) + \frac{1}{2}(b_{ij} - b_{ji}) \\
0 &= m_{in}\ddot{q}^i + (b_{ni} - b_{in})\,\dot{q}^i + k_{in}e^{q^i+q^n} \\
&= 2(\text{anti-symmetric part of } b_{ni}) + \dots
\end{aligned}$$

$\implies$ the equations of motion depend only on the anti-symmetric part of b_{ij}.

Problem 3

(a)

$$
\begin{aligned}
p_n &= \frac{\partial L}{\partial v_n} \\
&= \frac{\partial}{\partial v_n}\left(-mc^2\left(1-\frac{{v_i}^2}{c^2}\right)^{\frac{1}{2}} + ex_iE_i + \frac{e}{2c}\epsilon_{ijk}B_ix_jv_k\right) \\
&= -\frac{1}{2}mc^2\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}}\left(0-\frac{2v_j\delta_{jn}}{c^2}\right) + 0 + \frac{e}{2c}\epsilon_{ijk}B_ix_j\delta_{kn} \\
p_n &= mv_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{e}{2c}\epsilon_{ijn}B_ix_j \\
p_3 &= mv_3\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{e}{2c}\epsilon_{ij3}B_ix_j \\
p_3 &= mv_3\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{e}{2c}(B_1x_2-B_2x_1)
\end{aligned}
$$

$$
\begin{aligned}
p_n &= mv_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{e}{2c}\epsilon_{ijn}B_ix_j \\
mv_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} &= p_n - \frac{e}{2c}\epsilon_{ijn}B_ix_j \\
v_n &= \frac{1}{m}\left(1-\frac{{v_i}^2}{c^2}\right)^{\frac{1}{2}}\left(p_n-\frac{e}{2c}\epsilon_{ijn}B_ix_j\right) \\
v_2 &= \frac{1}{m}\left(1-\frac{{v_i}^2}{c^2}\right)^{\frac{1}{2}}\left(p_2-\frac{e}{2c}\epsilon_{ij2}B_ix_j\right) \\
v_2 &= \frac{1}{m}\left(1-\frac{{v_i}^2}{c^2}\right)^{\frac{1}{2}}\left(p_2-\frac{e}{2c}(B_3x_1-B_1x_3)\right)
\end{aligned}
$$

(b)

$$\frac{d}{dt}\frac{\partial L}{\partial v_n} = \frac{\partial L}{\partial x_n}$$

$$\begin{aligned}
\frac{\partial L}{\partial v_n} &= mv_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{e}{2c}\epsilon_{ijn}B_i x_j \\
\frac{d}{dt}\frac{\partial L}{\partial v_n} &= ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + mv_n\left(-\frac{1}{2}\right)\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}}\left(0-\frac{2v_i a_i}{c^2}\right) + \frac{e}{2c}\epsilon_{ijn}B_i v_j \\
&= ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{mv_n v_i a_i}{c^2}\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} + \frac{e}{2c}\epsilon_{ijn}B_i v_j
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial x_n} &= \frac{\partial}{\partial x_n}\left(-mc^2\left(1-\frac{{v_i}^2}{c^2}\right)^{\frac{1}{2}} + ex_iE_i + \frac{e}{2c}\epsilon_{ijk}B_i x_j v_k\right) \\
&= 0 + e\delta_{in}E_i + \frac{e}{2c}\epsilon_{ijk}B_i\delta_{jn}v_k \\
&= eE_n + \frac{e}{2c}\epsilon_{ink}B_i v_k
\end{aligned}$$

$$\begin{aligned}
\implies ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{mv_n v_i a_i}{c^2}\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} + \frac{e}{2c}\epsilon_{ijn}B_i v_j &= eE_n + \frac{e}{2c}\epsilon inkB_i v_k \\
ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{mv_n v_i a_i}{c^2}\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} &= eE_n + \frac{e}{2c}\epsilon_{ink}B_i v_k - \frac{e}{2c}\epsilon_{ijn}B_i v_j \\
&= eE_n + \frac{e}{2c}(\epsilon_{ink}B_i v_k + \epsilon_{ink}B_i v_k) \\
ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{mv_n v_i a_i}{c^2}\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} &= eE_n + \frac{e}{c}\epsilon_{ink}B_i v_k
\end{aligned}$$

(c)

$$\vec{E} = 0 \implies ma_n\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{mv_n v_i a_i}{c^2}\left(1-\frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} = \frac{e}{c}\epsilon_{ink}B_i v_k$$

We want to get rid of the term with B_i. To do this we multiply the expression by v_n.

$$\begin{aligned}
\frac{e}{c}\epsilon_{ink}B_i v_n v_k &= \frac{e}{c}(B_1v_2v_3 - B_1v_3v_2 + B_2v_3v_1 - B_2v_1v_3 + B_3v_1v_2 - B_3v_2v_1) \\
&= 0
\end{aligned}$$

$$\implies mv_n a_n \left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{m{v_n}^2 v_i a_i}{c^2}\left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}} = 0$$

$$\frac{m}{2}\left(\left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{{v_i}^2}{c^2}\left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}}\right) 2v_n a_n = 0$$

$$\frac{m}{2}\left(\left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{{v_i}^2}{c^2}\left(1 - \frac{{v_i}^2}{c^2}\right)^{-\frac{3}{2}}\right) \frac{d}{dt}{v_n}^2 = 0$$

Every term in this expression except $\frac{d}{dt}{v_n}^2$ must be greater than 0 $\implies \frac{d}{dt}{v_n}^2 = 0$ thus the rate of change of the magnitude of the velocity is 0, i.e. the speed is constant.