MAU23401: Advanced Classical Mechanics II
Homework 8 due 02/04/2021

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Problem 1

(i)

$$\begin{aligned}
H &= p_i v_i - L \\
&= p_i v_i + mc^2\sqrt{1-\frac{v_i^2}{c^2}} - \frac{e}{2c}\epsilon_{ijk}B_i x_j v_k \\
&= \vec{p}\cdot\frac{c\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}} + \frac{m^2c^3}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}} \\
&\quad - \frac{e}{2c}\epsilon_{ijk}B_i x_j \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(p_k - \frac{e}{2c}\epsilon_{klm}B_l x_m\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(\vec{p}^2 - \frac{e}{2c}\vec{p}\cdot\left(\vec{B}\times\vec{r}\right) + m^2c^2 - \frac{e}{2c}\vec{p}\cdot\left(\vec{B}\times\vec{r}\right) + \frac{e^2}{4c^2}\epsilon_{ijk}B_i x_j \epsilon_{klm}B_l x_m\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(m^2c^2 + \vec{p}^2 - 2\vec{p}\cdot\left(\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right) + \left(\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(m^2c^2 + \left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)
\end{aligned}$$

$$H = c\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}$$

Hamiltonian has no explicit time dependence and so $E = H$.

$$\vec{F} = -\frac{\partial U}{\partial \vec{r}} = \frac{e}{2c}\vec{v}\times\vec{B} \perp \vec{v} \implies v^2 = \text{constant}$$

(ii)

$$
\begin{aligned}
H &= c\sqrt{m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2} \\
&= mc^2\sqrt{1 + \frac{\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}{m^2c^2}} \\
\vec{B}\times\vec{r} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & B \\ q_x & q_y & q_z \end{vmatrix} \\
&= -Bq_y\hat{\imath} + Bq_x\hat{\jmath}
\end{aligned}
$$

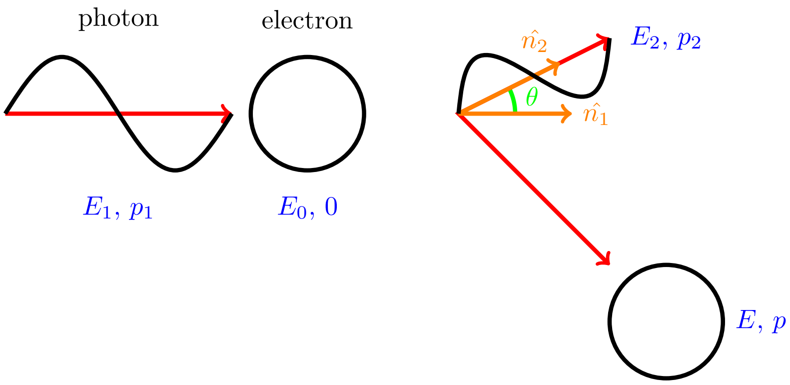
$$
\begin{aligned}
\dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{r}} \\
\dot{\vec{p}} &= \frac{e\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)\times\vec{B}}{2mc\sqrt{1 + \frac{\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}{m^2c^2}}} \\
&= \frac{Be}{2mc\sqrt{1 + \frac{\left(p_x + \frac{Be}{2c}q_y\right)^2 + \left(p_y - \frac{Be}{2c}q_x\right)^2 + p_z^2}{m^2c^2}}}\left(\left(p_y - \frac{Be}{2c}q_x\right)\hat{\imath} + \left(-p_x - \frac{Be}{2c}q_y\right)\hat{\jmath}\right) \\
\dot{\vec{r}} &= \frac{\partial H}{\partial \vec{p}} \\
\dot{\vec{r}} &= \frac{\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)}{m\sqrt{1 + \frac{\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}{m^2c^2}}} \\
&= \frac{1}{m\sqrt{1 + \frac{\left(p_x + \frac{Be}{2c}q_y\right)^2 + \left(p_y - \frac{Be}{2c}q_x\right)^2 + p_z^2}{m^2c^2}}}\left(\left(p_x + \frac{Be}{2c}q_y\right)\hat{\imath} + \left(p_y - \frac{Be}{2c}q_x\right)\hat{\jmath} + p_z\hat{k}\right)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\dot{\vec{p}} &= \vec{F} \\
&= \frac{e}{2c}\vec{v} \times \vec{B} \\
&= \frac{Be}{2c}\left(v_y \hat{\imath} - v_x \hat{\jmath}\right) \\
\implies \vec{p} &= \frac{Be}{2c}\left(\left(q_y + c_x\right)\hat{\imath} + \left(-q_x + c_x\right)\hat{\jmath} + c_z\hat{k}\right) \\
&\approx \frac{Be}{2c}\left(q_y\hat{\imath} - q_x\hat{\jmath} + c\hat{k}\right) \\
\dot{\vec{r}} &= \frac{1}{m\sqrt{1 + \frac{\left(p_x + \frac{Be}{2c}q_y\right)^2 + \left(p_y - \frac{Be}{2c}q_x\right)^2 + p_z^2}{m^2c^2}}}\left(\left(p_x + \frac{Be}{2c}q_y\right)\hat{\imath} + \left(p_y - \frac{Be}{2c}q_x\right)\hat{\jmath} + p_z\hat{k}\right) \\
&= \frac{1}{m\sqrt{1 + \frac{1}{m^2c^2}\left(\left(\frac{Be}{c}q_y\right)^2 + \left(-\frac{Be}{c}q_x\right)^2 + c^2\right)}}\left(\frac{Be}{c}q_y\hat{\imath} - \frac{Be}{c}q_x\hat{\jmath} + c\hat{k}\right) \\
&= \left(Kq_y\hat{\imath} - Kq_x\hat{\jmath} + c\hat{k}\right) \\
\implies \vec{r} &= \left(\kappa\sin\phi\,\hat{\imath} + \kappa\cos\phi\,\hat{\jmath} + ct\,\hat{k}\right) \\
K &= \frac{Be}{mc\sqrt{1 + \frac{1}{m^2c^2}\left(\kappa^2\sin^2\phi + \kappa^2\cos^2\phi + c^2\right)}} \\
&= \text{constant} \implies \kappa = \text{constant} \\
\vec{r} &= \kappa\left(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} + Ct\,\hat{k}\right)
\end{aligned}
$$

If $C = 0$, then the motion is simply circular in the xy-plane. If $C \neq 0$, then the motion is a helix travelling in the $\hat{k}$ or $-\hat{k}$ direction, depending on the sign of C.

Problem 2



$$\text{Energy conservation} \implies E_1 + E_0 = E + E_2$$
$$E = E_1 - E_2 + mc^2 \tag{1}$$

$$\text{Momentum conservation} \implies \frac{E_1}{c}\hat{n}_1 = \frac{E_2}{c}\hat{n}_2 + \vec{p}$$
$$\vec{p}\cdot\vec{p} = \frac{1}{c}(E_1\hat{n}_1 - E_2\hat{n}_2)\cdot\frac{1}{c}(E_1\hat{n}_1 - E_2\hat{n}_2)$$
$$p^2 = \frac{1}{c^2}(E_1^2 - 2E_1E_2\cos(\theta) + E_2^2) \tag{2}$$

$$\text{Energy-momentum invariant} \implies E^2 - p^2c^2 = m^2c^4 \tag{3}$$

$$(1),\ (2)\ \&\ (3) \implies m^2c^4 = (E_1 - E_2 + mc^2)^2 - \frac{c^2}{c^2}(E_1^2 - 2E_1E_2\cos(\theta) + E_2^2)$$
$$m^2c^4 = E_1^2 - 2E_1E_2 + E_2^2 + 2mc^2(E_1 - E_2) + m^2c^4$$
$$- E_1^2 + 2E_1E_2\cos(\theta) - E_2^2$$
$$0 = -2E_1E_2 + 2mc^2(E_1 - E_2) + 2E_1E_2\cos(\theta)$$
$$2(E_1 - E_2)mc^2 = 2E_1E_2(1 - \cos(\theta))$$
$$\frac{E_1 - E_2}{E_1E_2} = \frac{1 - \cos(\theta)}{mc^2}$$
$$\frac{1}{E_2} - \frac{1}{E_1} = \frac{1 - \cos(\theta)}{mc^2}$$
$$\frac{1}{hc \div \lambda_2} - \frac{1}{hc \div \lambda_1} = \frac{1 - \cos(\theta)}{mc^2}$$
$$\lambda_2 - \lambda_1 = \frac{h(1 - \cos(\theta))}{mc} \tag{4}$$

$\lambda_1 \equiv \lambda$, $\lambda_2 \equiv \lambda'$, $E_1 \equiv h\nu$, $E_2 \equiv h\nu'$, $E \equiv T + mc^2$

$$
\begin{aligned}
(4) \implies \lambda' - \lambda &= \lambda_c(1 - \cos\theta) \\
&= 2\lambda_c \sin^2\frac{\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
(1) \implies T + mc^2 &= h\nu - h\nu' + mc^2 \\
T &= h\nu\left(1 - \frac{\nu'}{\nu}\right) \\
&= h\nu\left(1 - \frac{\frac{c}{\lambda'}}{\frac{c}{\lambda}}\right) \\
&= h\nu\left(1 - \frac{\lambda}{\lambda'}\right) \\
&= h\nu\left(\frac{\lambda' - \lambda}{\lambda'}\right) \\
&= h\nu\left(\frac{2\lambda_c \sin^2\frac{\theta}{2}}{\lambda + 2\lambda_c \sin^2\frac{\theta}{2}}\right) \\
&= h\nu\frac{2\frac{\lambda_c}{\lambda}\sin^2\frac{\theta}{2}}{1 + 2\frac{\lambda_c}{\lambda}\sin^2\frac{\theta}{2}}
\end{aligned}
$$