MAU23401: Advanced Classical Mechanics II
Homework 7 due 26/03/2021

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Problem 1

$$v = \frac{v' + V}{1 + \frac{V}{c^2} v'}$$

Say $|v'| < c$ and $|V| < c$.

$$\begin{aligned} c - v &= c - \frac{v' + V}{1 + \frac{V}{c^2} v'} \\ &= \frac{c + \frac{V}{c} v' - v' - V}{1 + \frac{V}{c^2} v'} \\ &= \frac{c\left(c^2 + Vv' - cv' - cV\right)}{c^2 + Vv'} \\ &= \frac{c(c - v')(c - V)}{c^2 + Vv'} \\ &> 0 \\ \implies c &> v \end{aligned}$$

$$\begin{aligned} v - c &= \frac{v' + V}{1 + \frac{V}{c^2} v'} - c \\ &= \frac{v' + V - c - \frac{V}{c} v'}{1 + \frac{V}{c^2} v'} \\ &= \frac{c\left(cv' + cV - c^2 - Vv'\right)}{c^2 + Vv'} \\ &= \frac{-c(c - v')(c - V)}{c^2 + Vv'} \\ &< 0 \\ \implies -c &< v \end{aligned}$$

Thus $-c < v < c$, i.e. $|v| < c$, and so if the speed of a particle is smaller than the light speed in one inertial frame of reference then it is smaller than the light speed in any other inertial frame, provided the inertial reference frames also travel slower than light.

Problem 2

$$x^0 = \gamma\left(x^{0\prime} + \frac{V}{c}x^{1\prime}\right) \qquad x^1 = \gamma\left(x^{1\prime} + \frac{V}{c}x^{0\prime}\right) \qquad x^2 = x^{2\prime} \qquad x^3 = x^{3\prime}$$

$$x^i = L^i{}_k x^{k\prime} \implies L^i_j = \begin{pmatrix} \gamma & \frac{\gamma V}{c} & 0 & 0 \\ \frac{\gamma V}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_i{}^j = \eta_{ik} L^k{}_l \eta^{lj}$$

$$L_0{}^0 = \eta_{0k} L^k{}_l \eta^{l0} = L^0{}_0 = \gamma \qquad L_0{}^1 = \eta_{0k} L^k{}_l \eta^{l1} = -L^0{}_1 = -\frac{\gamma V}{c}$$

$$L_1{}^0 = \eta_{1k} L^k{}_l \eta^{l0} = -L^0{}_1 = -\frac{\gamma V}{c} \qquad L_1{}^1 = \eta_{1k} L^k{}_l \eta^{l1} = L^1{}_1 = \gamma$$

$$L_2{}^2 = \eta_{2k} L^k{}_l \eta^{l2} = L^2{}_2 = 1 \qquad L_3{}^3 = \eta_{3k} L^k{}_l \eta^{l3} = L^3{}_3 = 1$$

$$F_{ij} = L_i{}^k L_j{}^l F'_{kl}$$

$$\begin{aligned} F_{01} = E_x &= L_0{}^k L_1{}^l F'_{kl} \\ &= L_0{}^0 L_1{}^0 F'_{00} + L_0{}^1 L_1{}^0 F'_{10} + L_0{}^0 L_1{}^1 F'_{01} + L_0{}^1 L_1{}^1 F'_{11} \\ &= 0 + \left(\frac{\gamma V}{c}\right)^2 (-E_x{}') + \gamma^2 E_x{}' + 0 \end{aligned}$$

$$E_x = E_x{}'$$

$$\begin{aligned} F_{02} = E_y &= L_0{}^k L_2{}^l F'_{kl} \\ &= L_0{}^0 L_2{}^2 F'_{02} + L_0{}^1 L_2{}^2 F'_{12} \end{aligned}$$

$$E_y = \gamma E_y{}' + \frac{\gamma V}{c} B_z$$

$$\begin{aligned} F_{03} = E_z &= L_0{}^k L_3{}^l F'_{kl} \\ &= L_0{}^0 L_3{}^3 F'_{03} + L_0{}^1 L_3{}^3 F'_{13} \end{aligned}$$

$$E_z = \gamma E_z{}' - \frac{\gamma V}{c} B_y$$

$$\begin{aligned} F_{32} = B_x &= L_3{}^k L_2{}^l F'_{kl} \\ &= L_3{}^3 L_2{}^2 F'_{32} \end{aligned}$$

$$B_x = B_x{}'$$

$$\begin{aligned} F_{13} = B_y &= L_1{}^k L_3{}^l F'_{kl} \\ &= L_1{}^0 L_3{}^3 F'_{03} + L_1{}^1 L_3{}^3 F'_{13} \end{aligned}$$

$$B_y = -\frac{\gamma V}{c} E_z + \gamma B_y$$

$$\begin{aligned} F_{21} = B_z &= L_2{}^k L_1{}^l F'_{kl} \\ &= L_2{}^2 L_1{}^0 F'_{20} + L_2{}^2 L_1{}^1 F'_{21} \end{aligned}$$

$$B_z = \frac{\gamma V}{c} E_y + \gamma B_z$$

Problem 3

(i)

$$
\begin{aligned}
S &\equiv \int_{t_1}^{t_2} L(x, v, t)\, dt \\
\implies L &= -mc^2\sqrt{1-\frac{v_i^2}{c^2}} + e\, x_i \boldsymbol{\varepsilon}_i \\
p_n &= \frac{\partial L}{\partial v_n} \\
&= -mc^2\left(1-\frac{v_i^2}{c^2}\right)^{-\frac{1}{2}}\left(\frac{1}{2}\right)\left(\frac{-2v_i\delta_n^i}{c^2}\right) \\
&= \frac{mv_n}{\sqrt{1-\frac{v_i^2}{c^2}}}
\end{aligned}
$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v_i^2}{c^2}}} \qquad\qquad p_3 = \frac{mv_3}{\sqrt{1-\frac{v_i^2}{c^2}}}$$

$$
\begin{aligned}
p_i^2 &= \frac{m^2v_i^2}{1-\frac{v_j^2}{c^2}} \\
\implies m^2v_i^2 &= p_i^2 - \frac{p_i^2v_j^2}{c^2} \\
v_i^2\left(m^2+\frac{p_j^2}{c^2}\right) &= p_i^2 \\
\frac{v_i^2}{c^2} &= \frac{p_i^2}{m^2c^2+p_j^2} \\
\implies \vec{p} &= \frac{m\vec{v}}{\sqrt{1-\frac{p_i^2}{m^2c^2+p_j^2}}} \\
\implies \vec{v} &= \frac{\vec{p}}{m}\sqrt{\frac{m^2c^2}{m^2c^2+p_i^2}}
\end{aligned}
$$

$$\vec{v} = \vec{p}c\sqrt{\frac{1}{m^2c^2+p_i^2}} \qquad\qquad v_2 = p_2c\sqrt{\frac{1}{m^2c^2+p_i^2}}$$

(ii)

$$
\begin{aligned}
\frac{\partial L}{\partial v_n} &= mv_n\left(1-\frac{v_i^2}{c^2}\right)^{-\frac{1}{2}} \\
\frac{d}{dt}\frac{\partial L}{\partial v_n} &= m\dot{v}_n mv_n\left(1-\frac{v_i^2}{c^2}\right)^{-\frac{1}{2}} + mv_n\left(1-\frac{v_i^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{1}{2}\right)\left(\frac{-2v_j\dot{v}_j}{c^2}\right) \\
&= \frac{m\dot{v}_n}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{mv_nv_j\dot{v}_j}{c^2\left(1-\frac{v_i^2}{c^2}\right)^{\frac{3}{2}}} \\
\frac{\partial L}{\partial x_n} &= e\,\boldsymbol{\varepsilon}_n
\end{aligned}
$$

$$\frac{m\dot{v}_n}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{m v_n v_j \dot{v}_j}{c^2\left(1-\frac{v_i^2}{c^2}\right)^{\frac{3}{2}}} = e\,\boldsymbol{\varepsilon}_n$$

(iii)

$$\begin{aligned}
H &= p_i v_i - L \\
&= p_i^2 c\sqrt{\frac{1}{m^2c^2+p_j^2}} + mc^2\sqrt{1-\frac{v_i^2}{c^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
&= p_i^2 c\sqrt{\frac{1}{m^2c^2+p_j^2}} + mc^2\sqrt{1-\frac{p_i^2}{m^2c^2+p_j^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
&= p_i^2 c\sqrt{\frac{1}{m^2c^2+p_j^2}} + mc^2\sqrt{\frac{m^2c^2}{m^2c^2+p_j^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
&= p_i^2 c\sqrt{\frac{1}{m^2c^2+p_j^2}} + m^2c^3\sqrt{\frac{1}{m^2c^2+p_j^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
&= \left(m^2c^3+p_i^2c\right)\sqrt{\frac{1}{m^2c^2+p_j^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
&= \left(m^2c^4+p_i^2c^2\right)\sqrt{\frac{1}{m^2c^4+p_j^2c^2}} - e\,x_i\boldsymbol{\varepsilon}_i \\
H &= \sqrt{m^2c^4+p_i^2c^2} - e\,x_i\boldsymbol{\varepsilon}_i
\end{aligned}$$

$$\dot{p}_n = -\frac{\partial H}{\partial x_n}$$

$$\dot{p}_n = e\,\boldsymbol{\varepsilon}_n$$

$$\begin{aligned}
\dot{x}_n &= \frac{\partial H}{\partial p_n} \\
&= \frac{1}{2}\left(m^2c^4+p_i^2c^2\right)^{-\frac{1}{2}}\left(2p_nc^2\right) \\
\dot{x}_n &= \frac{p_nc^2}{\sqrt{m^2c^4+p_i^2c^2}}
\end{aligned}$$

(iv)

$$\dot{p}_x = e\,\boldsymbol{\varepsilon} \implies p_x = e\,\boldsymbol{\varepsilon}\,t \qquad \dot{p}_y = 0 \implies p_y = p_0 \qquad \dot{p}_z = 0 \implies p_z = 0$$

$$\dot{x} = \frac{e\,\boldsymbol{\varepsilon}\,t\,c^2}{\sqrt{m^2c^4+e^2\boldsymbol{\varepsilon}^2t^2c^2+p_0^2c^2}} \qquad \dot{y} = \frac{p_0c^2}{\sqrt{m^2c^4+e^2\boldsymbol{\varepsilon}^2t^2c^2+p_0^2c^2}} \qquad \dot{z} = 0 \implies z = 0$$

Labelling $E_0 = m^2c^4 + p_0^2c^2$,

$$x(t) = \frac{\sqrt{E_0^2+e^2\boldsymbol{\varepsilon}^2c^2t^2} - E_0}{e\,\boldsymbol{\varepsilon}} \qquad y(t) = \frac{p_0c}{e\,\boldsymbol{\varepsilon}}\operatorname{arcsinh}\left(\frac{e\,\boldsymbol{\varepsilon}\,c\,t}{E_0}\right)$$

$$\begin{aligned}
y(x) &= \frac{p_0c}{e\,\boldsymbol{\varepsilon}}\operatorname{arcsinh}\left(\frac{\sqrt{2E_0e\,\boldsymbol{\varepsilon}\,x+e^2\boldsymbol{\varepsilon}^2x^2}}{E_0}\right) \\
&\sim \operatorname{arcsinh}\left(\sqrt{x^2+2x}\right)
\end{aligned}$$

```
x[t_] =
 Assuming[{m > 0 && ϵ ≥ 0}, Integrate[e ϵ t0 C^2 / Sqrt[m^2 C^4 + p_0^2 C^2 + e^2 ϵ^2 t0^2 C^2], {t0, 0, t}]] // FullSimplify

y[t_] = Assuming[{m > 0 && ϵ ≥ 0}, Integrate[p_0 C^2 / Sqrt[m^2 C^4 + p_0^2 C^2 + e^2 ϵ^2 t0^2 C^2], {t0, 0, t}]] //
  FullSimplify
```

$$\frac{\sqrt{C^2\left(C^2 m^2+p_0^2\right)}\left(-1+\sqrt{1+\frac{e^2 t^2 \epsilon^2}{C^2 m^2+p_0^2}}\right)}{e\,\epsilon} \quad \text{if} \quad \textit{condition}$$

$$\frac{\text{ArcSinh}\left[\frac{e\,t\,\epsilon}{\sqrt{C^2 m^2+p_0^2}}\right] p_0 \sqrt{C^2\left(C^2 m^2+p_0^2\right)}}{e\,\epsilon\,\sqrt{C^2 m^2+p_0^2}} \quad \text{if} \quad \textit{condition}$$

```
Clear[E0, e, ϵ, t, p];
Solve[(Sqrt[E0^2 + e^2 ϵ^2 C^2 t^2] - E0)/(e ϵ) == x, t]
```

$$\left\{\left\{t \to -\frac{\sqrt{x}\,\sqrt{2\,E0+e\,x\,\epsilon}}{C\,\sqrt{e}\,\sqrt{\epsilon}}\right\}, \left\{t \to \frac{\sqrt{x}\,\sqrt{2\,E0+e\,x\,\epsilon}}{C\,\sqrt{e}\,\sqrt{\epsilon}}\right\}\right\}$$

```
t = Sqrt[x] Sqrt[2 E0 + e x ϵ]/(C Sqrt[e] Sqrt[ϵ]);
Clear[E0, e, ϵ, p, x];
y[t_] = (p C)/(e ϵ) ArcSinh[(e ϵ C t)/E0]
```

$$\frac{C\,p\,\text{ArcSinh}\left[\frac{\sqrt{e}\,\sqrt{x}\,\sqrt{\epsilon}\,\sqrt{2\,E0+e\,x\,\epsilon}}{E0}\right]}{e\,\epsilon}$$

```
E0 = e = ϵ = p = 1;
y[t_] = (p C)/(e ϵ) ArcSinh[(e ϵ C t)/E0]
```

$$C\,\text{ArcSinh}\left[\sqrt{x}\,\sqrt{2+x}\right]$$

```
Plot[ArcSinh[√(x² + 2 x)], {x, 0, 1000}]
```