# MAU23401: Advanced Classical Mechanics II
# Homework 5 due 05/03/2021

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I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

## Problem 1.

### (i)

$$\begin{aligned} H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = E \\ \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega^2}} = 1 \end{aligned}$$

The equation for an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the area for the ellipse is $A = \pi ab$. We can thus consider the ellipse in the phase space with $a = \sqrt{2mE}$ and $b = \sqrt{\frac{2E}{m\omega^2}}$.

$$\begin{aligned} A &= \pi ab \\ &= \pi\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}} \\ &= 2\pi\frac{E}{\omega} \\ I &= \frac{A}{2\pi} \\ &= \frac{E}{\omega} \\ \implies E &= I\omega \\ &= I\sqrt{\frac{g}{L}} \\ E &\sim \frac{1}{\sqrt{L}} \end{aligned}$$

The change in length of the pendulum is equal to the change in position of the ring.

### (ii)

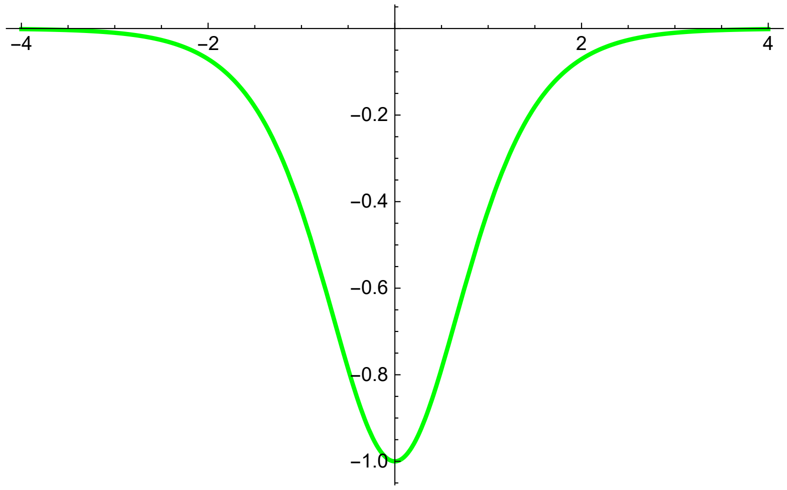
The amplitude of the pendulum will be attained when there is no motion, i.e. when $p = 0$.

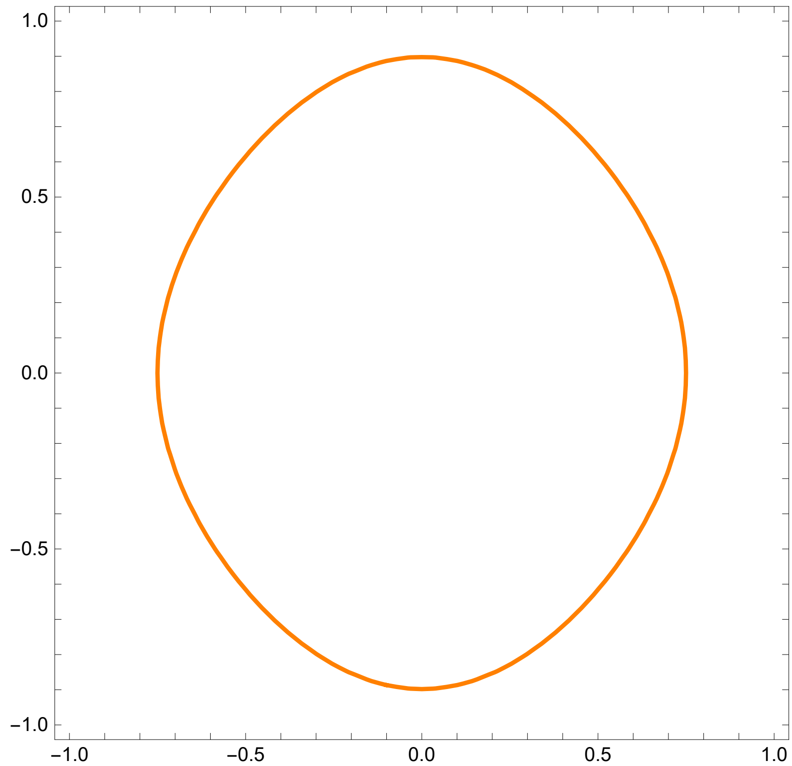
$$\begin{aligned} \implies \frac{q_{\text{max}}^2}{\frac{2E}{m\omega^2}} &= 1 \\ q_{\text{max}} &= \sqrt{\frac{2E}{m\omega^2}} \\ &= \sqrt{\frac{2I}{m}\sqrt{\frac{L}{g}}} \\ q_{\text{max}} &\sim L^{\frac{1}{4}} \end{aligned}$$

## Problem 2.

(i)

```
U[x_] = -V/Cosh[a x]^2;
H[x_, p_] = p^2/(2 m) - V/Cosh[a x]^2;
m = 1;
a = 1;
V = 1;
e = U[0.75];
Plot[U[x], {x, -4, 4}, PlotStyle → {Green, Thick}]
ContourPlot[H[x, p] == e, {x, -1, 1}, {p, -1, 1}, ContourStyle → {Orange, Thick}]
Clear[m, a, V, e]
```



(ii)

$$
\begin{aligned}
H &= \frac{p^2}{2m} - \frac{V}{\cosh^2 ax} = E \\
p &= \sqrt{2mE + \frac{2mV}{\cosh^2 ax}} \\
p &= 0 \implies \cosh^2 ax = -\frac{V}{E} \\
A &= 2\int_{x_1}^{x_2} p\, dx \\
&= 2\int_{x_1}^{x_2} \sqrt{2mE + \frac{2mV}{\cosh^2 ax}}\, dx \\
x &= \frac{1}{a}\operatorname{arcsinh} y,\ \cosh^2 ax = y^2+1,\ \text{bounds } y^2 = \frac{-V-E}{E},\ dx = \frac{dy}{a\sqrt{y^2+1}} \\
\implies A &= 2\int_{y_1}^{y_2} \frac{1}{a\sqrt{y^2+1}}\sqrt{2mE + \frac{2mV}{y^2+1}}\, dy \\
&= \frac{2\pi\sqrt{2}}{a}\left(\sqrt{mV} - \sqrt{-mE}\right) \\
I &= \frac{A}{2\pi} \\
I &= \frac{\sqrt{2}}{a}\left(\sqrt{mV} - \sqrt{-mE}\right)
\end{aligned}
$$

```
Assuming[{V > 0 && m > 0 && e < 0},
  Integrate[2/(a Sqrt[y^2 + 1]) Sqrt[2 m e + (2 m V)/(y^2 + 1)], {y, -Sqrt[(-V - e)/e], Sqrt[(-V - e)/e]}]] // FullSimplify
```

$$\boxed{\frac{2\sqrt{2}\,\pi\left(-\sqrt{-e\,m} + \sqrt{m\,V}\right)}{a} \quad \text{if } e + V > 0}$$

(iii)

$$\begin{aligned}
I &= \frac{2}{a}\left(\sqrt{mV} - \sqrt{-mE}\right) \\
\implies E &= -\frac{1}{m}\left(\sqrt{mV} - \frac{Ia}{\sqrt{2}}\right)^2 \\
\omega &= \frac{\partial E}{\partial I} \\
&= \frac{a\sqrt{2}}{m}\left(\sqrt{mV} - \frac{Ia}{\sqrt{2}}\right) \\
\omega(E) &= a\sqrt{-\frac{2E}{m}}
\end{aligned}$$

$$\mathtt{D}\left[\frac{-1}{m}\left(\sqrt{m\,V} - \frac{i\,a}{\sqrt{2}}\right)^2,\ i\right]$$

$$\frac{\sqrt{2}\ a\left(-\frac{a\,i}{\sqrt{2}} + \sqrt{m\,V}\right)}{m}$$

$$i = \frac{\sqrt{2}}{a}\left(\sqrt{m\,V} - \sqrt{-m\,e}\right);$$

$$\omega = \frac{\sqrt{2}\ a\left(-\frac{a\,i}{\sqrt{2}} + \sqrt{m\,V}\right)}{m}\ \mathtt{//\ FullSimplify}$$

$$\frac{\sqrt{2}\ a\ \sqrt{-e\,m}}{m}$$

**(iv)**

$$
\begin{aligned}
\frac{p^2}{2m} - \frac{V}{\cosh^2 ax} &= E \\
\implies \frac{p^2}{2m} - V + Va^2x^2 &\approx E \\
\frac{p^2}{2m(E+V)} + \frac{x^2}{\frac{E+V}{Va^2}} &= 1
\end{aligned}
$$

$$
\begin{aligned}
A &= \pi\sqrt{2m(E+V)\left(\frac{E+V}{Va^2}\right)} \\
I &= \frac{A}{2\pi} \\
&= \frac{E+V}{a}\sqrt{\frac{m}{2V}} \\
\implies E &= aI\sqrt{\frac{2V}{m}} - V \\
\omega &= \frac{\partial E}{\partial I} \\
&= a\sqrt{\frac{2V}{m}} \\
E &= I\omega - V \\
E &\sim \omega \implies \text{harmonic oscillator}
\end{aligned}
$$

```
Series[V/Cosh[a x]^2, {x, 0, 2}]
```

$$V - a^2 V x^2 + O[x]^3$$

**(v)**

$$I(E) = \frac{2}{a}\left(\sqrt{mV} - \sqrt{-mE}\right)$$
$$I(x,p) = \frac{1}{a}\left(\sqrt{2mV} - \sqrt{2mV\,\mathrm{sech}^2\,ax - p^2}\right)$$

$$\frac{d\varphi}{dt} = \omega$$
$$\varphi = \omega t + \varphi_0$$
$$= at\sqrt{-\frac{2E}{m}} + \varphi_0$$

$$\mathbf{e} = \frac{p^2}{2\,m} - \frac{V}{\mathrm{Cosh}[a\,x]^2};$$

$$\mathbf{i} = \frac{\sqrt{2}}{a}\left(\sqrt{m\,V} - \sqrt{-m\,e}\right) \mathbin{//} \mathrm{FullSimplify}$$

$$\frac{\sqrt{2}\ \sqrt{m\,V} - \sqrt{-p^2 + 2\,m\,V\,\mathrm{Sech}[a\,x]^2}}{a}$$