MAU23401: Advanced Classical Mechanics II
Homework 5 due 05/03/2021

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Problem 1.

(i)

$$\begin{aligned}
x(\rho,\phi,z) &= \rho\cos\phi \\
y(\rho,\phi,z) &= \rho\sin\phi \\
z(\rho,\phi,z) &= z \\
r(\rho,\phi,z) &= \sqrt{(z-a)^2+\rho^2} \\
L &= K-U \\
L(x,y,z,r) &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-\frac{kqQ}{r}-mgz \\
L(\rho,\phi,z) &= \frac{m}{2}\left(\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{z}^2\right)-\frac{kqQ}{\sqrt{(z-a)^2+\rho^2}}-mgz
\end{aligned}$$

```
Clear[z, ξ, η, x, y, z, r, ρ, ϕ]
x[t_] = ρ[t] Cos[ϕ[t]];
y[t_] = ρ[t] Sin[ϕ[t]];
r[t] = √((z[t] - a)^2 + ρ[t]^2);
L = m/2 (D[x[t], t]^2 + D[y[t], t]^2 + D[z[t], t]^2) - k q Q/r[t] - m g z[t] // FullSimplify
```

$$-\frac{k\,q\,Q}{\sqrt{(a-z[t])^2+\rho[t]^2}}+\frac{1}{2}m\left(-2\,g\,z[t]+z'[t]^2+\rho'[t]^2+\rho[t]^2\,\phi'[t]^2\right)$$

(ii)

$$\begin{aligned}
\xi-\eta &= z+a \\
z(\xi,\eta) &= \xi-\eta-a \\
\xi+\eta &= r \\
\sqrt{(z-a)^2+\rho^2} &= \xi+\eta \\
\rho^2 &= (\xi+\eta)^2-(\xi-\eta-a-a)^2 \\
\rho(\xi,\eta) &= 2\sqrt{(\xi-a)(\eta+a)}
\end{aligned}$$

```
√((ξ + η)^2 - (ξ - η - 2 a)^2) // FullSimplify
```

$$2\sqrt{-(a+\eta)(a-\xi)}$$

(iii)

$$L(\rho,\phi,z)=\frac{m}{2}\left(\dot\rho^2+\rho^2\dot\phi^2+\dot z^2\right)-\frac{kqQ}{\sqrt{(z-a)^2+\rho^2}}-mgz$$

$$L(\xi,\eta,\phi)=\frac{m}{2}\left(\frac{\dot\xi^2(\xi+\eta)}{\xi-a}+\frac{\dot\eta^2(\xi+\eta)}{\eta+a}+4\dot\phi^2(\xi-a)(\eta+a)\right)-\frac{kqQ}{\xi+\eta}-mg(\xi-\eta-a)$$

```
ρ[t_] = 2 √((ξ[t] - a) (η[t] + a));
z[t_] = ξ[t] - η[t] - a;
L // FullSimplify
```

$$-\frac{k\,q\,Q}{\sqrt{(\eta[t]+\xi[t])^2}}+\frac{1}{2}m\left(2g\,(a+\eta[t]-\xi[t])+(\eta'[t]-\xi'[t])^2+\frac{((-a+\xi[t])\,\eta'[t]+(a+\eta[t])\,\xi'[t])^2}{(a+\eta[t])\,(-a+\xi[t])}+4\,(a+\eta[t])\,(-a+\xi[t])\,\phi'[t]^2\right)$$

(iv)

$$\begin{aligned}H&=p_i\dot q^i-L\\&=\frac{\partial L}{\partial\dot q^i}\dot q^i-L\\&=\frac{m}{2}\left(\frac{\dot\xi^2(\xi+\eta)}{\xi-a}+\frac{\dot\eta^2(\xi+\eta)}{\eta+a}+4\dot\phi^2(\xi-a)(\eta+a)\right)+\frac{kqQ}{\xi+\eta}+mg(\xi-\eta-a)\end{aligned}$$

$$p_\xi=\frac{\partial L}{\partial\dot\xi}=\frac{m\dot\xi\,(\xi+\eta)}{\xi-a}\qquad\Longrightarrow\frac{\dot\xi^2\,(\xi+\eta)}{\xi-a}=\frac{p_\xi^2\,(\xi-a)}{m^2\,(\xi+\eta)}$$

$$p_\eta=\frac{\partial L}{\partial\dot\eta}=\frac{m\dot\eta\,(\xi+\eta)}{\eta+a}\qquad\Longrightarrow\frac{\dot\eta^2\,(\xi+\eta)}{\eta+a}=\frac{p_\eta^2\,(\eta+a)}{m^2\,(\xi+\eta)}$$

$$p_\phi=\frac{\partial L}{\partial\dot\phi}=4m\,\dot\phi\,(\xi-a)\,(\eta+a)\qquad\Longrightarrow 4\dot\phi^2\,(\xi-a)\,(\eta+a)=\frac{p_\phi^2}{4m^2\,(\xi-a)\,(\eta+a)}$$

$$H(\xi,\eta,\phi,p_\xi,p_\eta,p_\phi)=\frac{p_\xi^2\,(\xi-a)}{2m\,(\xi+\eta)}+\frac{p_\eta^2\,(\eta+a)}{2m\,(\xi+\eta)}+\frac{p_\phi^2}{8m\,(\xi-a)\,(\eta+a)}+\frac{kqQ}{\xi+\eta}+mg\,(\xi-\eta-a)$$

```
H = D[L, ξ′[t]] ξ′[t] + D[L, η′[t]] η′[t] + D[L, ϕ′[t]] ϕ′[t] - L // FullSimplify
```

$$-a\,g\,m + \frac{k\,q\,Q}{\sqrt{(\eta[t]+\xi[t])^2}} + \left(m\left((-a+\xi[t])\left(2g\,(a+\eta[t])\,(\eta[t]-\xi[t]) - (\eta[t]+\xi[t])\,\eta'[t]^2\right) - (a+\eta[t])\,(\eta[t]+\xi[t])\,\xi'[t]^2 - 4\,(a+\eta[t])^2\,(a-\xi[t])^2\,\phi'[t]^2\right)\right) \Big/ \left(2\,(a+\eta[t])\,(a-\xi[t])\right)$$

$$K = \frac{1}{2} m \left(\frac{(\eta[t]+\xi[t])\,\eta'[t]^2}{a+\eta[t]} - \frac{(\eta[t]+\xi[t])\,\xi'[t]^2}{a-\xi[t]} - 4\,(a+\eta[t])\,(a-\xi[t])\,\phi'[t]^2\right);$$

$$U = -\left(g\,m\,(a+\eta[t]-\xi[t]) - \frac{k\,q\,Q}{\sqrt{(\eta[t]+\xi[t])^2}}\right);$$

```
L - K + U // FullSimplify
H - K - U // FullSimplify
```

0

0

```
D[L, ξ′[t]] // FullSimplify
D[L, η′[t]] // FullSimplify
D[L, ϕ′[t]] // FullSimplify
```

$$-\frac{m\,(\eta[t]+\xi[t])\,\xi'[t]}{a-\xi[t]}$$

$$\frac{m\,(\eta[t]+\xi[t])\,\eta'[t]}{a+\eta[t]}$$

$$4\,m\,(a+\eta[t])\,(-a+\xi[t])\,\phi'[t]$$

$$H = \frac{m}{2}\left(\frac{p_\xi{}^2\,(\xi-a)}{m^2\,(\xi+\eta)} + \frac{p_\eta{}^2\,(\eta+a)}{m^2\,(\xi+\eta)} + \frac{p_\phi{}^2}{4\,m^2\,(\xi-a)\,(\eta+a)}\right) + \frac{k\,q\,Q}{\xi+\eta} + m\,g\,(\xi-\eta-a) \text{ // FullSimplify}$$

$$-g\,m\,(a+\eta-\xi) + \frac{k\,q\,Q}{\eta+\xi} + \frac{\frac{4\,(a+\eta)\,p_\eta^2 + 4\,(-a+\xi)\,p_\xi^2}{\eta+\xi} + \frac{p_\phi^2}{(a+\eta)\,(-a+\xi)}}{8\,m}$$

(v)

$$\Phi\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}\right) = \frac{\partial S}{\partial t} + H(\xi, \eta, \phi, p_\xi, p_\eta, p_\phi) = 0$$

$$f_t\left(t, \frac{\partial S}{\partial t}\right) = \frac{\partial S}{\partial t}$$

$$\implies S_t = -Et$$

$$-E + H = 0$$

$$\frac{p_\xi^2\,(\xi-a)}{2m\,(\xi+\eta)} + \frac{p_\eta^2\,(\eta+a)}{2m\,(\xi+\eta)} + \frac{p_\phi^2}{8m\,(\xi-a)\,(\eta+a)} + \frac{kqQ}{\xi+\eta} + mg\,(\xi-\eta-a) = E$$

$$\phi \text{ is cyclic} \implies p_\phi = \alpha_\phi,\; S_\phi = \alpha_\phi \phi$$

$$\frac{p_\xi^2(\xi-a)}{2m(\xi+\eta)}+\frac{p_\eta^2(\eta+a)}{2m(\xi+\eta)}+\frac{\alpha_\phi^2}{8m(\xi-a)(\eta+a)}+\frac{kqQ}{\xi+\eta}+mg(\xi-\eta-a)=E$$

$$\left(mg\xi^2-mga\xi+\frac{p_\xi^2(\xi-a)}{2m}+\frac{\alpha_\phi^2}{\xi-a}-E\xi\right)+\left(-mg\eta^2-mga\eta+\frac{p_\eta^2(\eta+a)}{2m}+\frac{\alpha_\phi^2}{\eta+a}-E\eta+kqQ\right)=0$$

$$f_\xi=mg\xi^2-mga\xi+\left(\frac{\partial S}{\partial\xi}\right)^2\frac{\xi-a}{2m}+\frac{\alpha_\phi^2}{\xi-a}-E\xi=\alpha_\xi$$

$$\frac{\partial S}{\partial\xi}=\sqrt{\frac{2m}{\xi-a}\left(\alpha_\xi-mg\xi^2+mga\xi-\frac{\alpha_\phi^2}{\xi-a}+E\xi\right)}$$

$$S_\xi=\int_{\xi_0}^{\xi}\sqrt{\frac{2m}{\xi'-a}\left(\alpha_\xi-mg\xi'^2+mga\xi'-\frac{\alpha_\phi^2}{\xi'-a}+E\xi'\right)}\,d\xi'$$

$$f_\eta=-mg\eta^2-mga\eta+\left(\frac{\partial S}{\partial\eta}\right)^2\frac{\eta+a}{2m}+\frac{\alpha_\phi^2}{\eta+a}-E\eta+\alpha_\xi+kqQ=\alpha_\eta$$

$$\text{Similarly, } S_\eta=\int_{\eta_0}^{\eta}\sqrt{\frac{2m}{\eta'+a}\left(\alpha_\eta+mg\eta'^2+mga\eta'-\frac{\alpha_\phi^2}{\eta'+a}+E\eta-\alpha_\xi-kqQ\right)}\,d\eta'$$

$$S(\xi,\eta,\phi,t)=-Et+a_\phi\phi+\int_{\xi_0}^{\xi}\sqrt{\frac{2m}{\xi'-a}\left(\alpha_\xi-mg\xi'^2+mga\xi'-\frac{\alpha_\phi^2}{\xi'-a}+E\xi'\right)}\,d\xi'$$
$$+\int_{\eta_0}^{\eta}\sqrt{\frac{2m}{\eta'+a}\left(\alpha_\eta+mg\eta'^2+mga\eta'-\frac{\alpha_\phi^2}{\eta'+a}+E\eta-\alpha_\xi-kqQ\right)}\,d\eta'$$

$$(\xi+\eta)\left(\frac{p_\xi{}^2(\xi-a)}{2m(\xi+\eta)}+\frac{p_\eta{}^2(\eta+a)}{2m(\xi+\eta)}+\frac{\alpha_\phi{}^2}{8m(\xi-a)(\eta+a)}+\frac{kqQ}{\xi+\eta}+mg(\xi-\eta-a)\right)\text{ // Expand // FullSimplify}$$

$$kqQ-gm(a+\eta-\xi)(\eta+\xi)+\frac{4(a+\eta)p_\eta^2+4(-a+\xi)p_\xi^2-\frac{(\eta+\xi)\alpha_\phi^2}{(a+\eta)(a-\xi)}}{8m}$$

Problem 2.

(i)

$$x(\rho,\phi,z)=\rho\cos\phi$$
$$y(\rho,\phi,z)=\rho\sin\phi$$
$$z(\rho,\phi,z)=z$$
$$r_1(\rho,\phi,z)=\sqrt{(z+a)^2+\rho^2}$$
$$r_2(\rho,\phi,z)=\sqrt{(z-a)^2+\rho^2}$$
$$L=K-U$$
$$L(x,y,z,r_1,r_2)=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{A}{r_1}+\frac{B}{r_2}$$
$$L(\rho,\phi,z)=\frac{m}{2}\left(\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{z}^2\right)+\frac{A}{\sqrt{(z+a)^2+\rho^2}}+\frac{B}{\sqrt{(z-a)^2+\rho^2}}$$

```
Clear[z, ξ, η, x, y, z, r, ρ, ϕ]
x[t_] = ρ[t] Cos[ϕ[t]];
y[t_] = ρ[t] Sin[ϕ[t]];
r1[t_] = Sqrt[(z[t] + a)^2 + ρ[t]^2];
r2[t_] = Sqrt[(z[t] - a)^2 + ρ[t]^2];
L = m/2 (D[x[t], t]^2 + D[y[t], t]^2 + D[z[t], t]^2) + A/r1[t] + B/r2[t] // FullSimplify
```

$$\frac{B}{\sqrt{(a-z[t])^2+\rho[t]^2}}+\frac{A}{\sqrt{(a+z[t])^2+\rho[t]^2}}+\frac{1}{2}m\left(z'[t]^2+\rho'[t]^2+\rho[t]^2\phi'[t]^2\right)$$

(ii)

$$\xi+\eta=r_1 \qquad \xi-\eta=r_2$$

$$z(\xi,\eta)=\frac{\xi\eta}{a} \qquad \rho(\xi,\eta)=\frac{\sqrt{(\xi^2-a^2)(a^2-\eta^2)}}{a}$$

$$\xi\in[a,\infty) \qquad \eta\in[-a,a]$$

```
Solve[ξ[t] == (r1[t] + r2[t])/2 && η[t] == (r1[t] - r2[t])/2, {ρ[t], z[t]}]
```

$$\left\{\left\{\rho[t]\to-\frac{i\sqrt{a^2-\eta[t]^2}\sqrt{a^2-\xi[t]^2}}{a},\ z[t]\to\frac{\eta[t]\times\xi[t]}{a}\right\},\left\{\rho[t]\to\frac{i\sqrt{a^2-\eta[t]^2}\sqrt{a^2-\xi[t]^2}}{a},\ z[t]\to\frac{\eta[t]\times\xi[t]}{a}\right\}\right\}$$

(iii)

$$L(\rho,\phi,z)=\frac{m}{2}\left(\dot\rho^2+\rho^2\dot\phi^2+\dot z^2\right)+\frac{A}{\sqrt{(z+a)^2+\rho^2}}+\frac{B}{\sqrt{(z-a)^2+\rho^2}}$$

$$L(\xi,\eta,\phi)=\frac{m}{2}\left(\frac{\dot\xi^2\left(\xi^2-\eta^2\right)}{\xi^2-a^2}+\frac{\dot\eta^2\left(\xi^2-\eta^2\right)}{a^2-\eta^2}+\frac{\dot\phi^2\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)}{a^2}\right)+\frac{A}{\xi+\eta}+\frac{B}{\xi-\eta}$$

```
ρ[t_] = Sqrt[(ξ[t]^2 - a^2) (a^2 - η[t]^2)]/a;
z[t_] = ξ[t] × η[t]/a;
L // FullSimplify
```

$$\frac{B}{\sqrt{(\eta[t]-\xi[t])^2}}+\frac{A}{\sqrt{(\eta[t]+\xi[t])^2}}-\frac{m\left(\eta[t]^2-\xi[t]^2\right)\left(\left(a^2-\xi[t]^2\right)\eta'[t]^2+\left(-a^2+\eta[t]^2\right)\xi'[t]^2\right)}{2\left(a^2-\eta[t]^2\right)\left(a^2-\xi[t]^2\right)}-\frac{m\left(a^2-\eta[t]^2\right)\left(a^2-\xi[t]^2\right)\phi'[t]^2}{2a^2}$$

(iv)

$$
\begin{aligned}
H &= \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \\
&= \frac{m}{2}\left(\frac{\dot{\xi}^2\left(\xi^2-\eta^2\right)}{\xi^2-a^2} + \frac{\dot{\eta}^2\left(\xi^2-\eta^2\right)}{a^2-\eta^2} + \frac{\dot{\phi}^2\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)}{a^2}\right) - \frac{A}{\xi+\eta} - \frac{B}{\xi-\eta}
\end{aligned}
$$

$$
\begin{aligned}
p_\xi &= \frac{\partial L}{\partial \dot{\xi}} = \frac{m\dot{\xi}\left(\xi^2-\eta^2\right)}{\xi^2-a^2} && \Longrightarrow \frac{\dot{\xi}^2\left(\xi^2-\eta^2\right)}{\xi^2-a^2} = \frac{p_\xi^2\left(\xi^2-a^2\right)}{m^2\left(\xi^2-\eta^2\right)} \\
p_\eta &= \frac{\partial L}{\partial \dot{\eta}} = \frac{m\dot{\eta}\left(\xi^2-\eta^2\right)}{a^2-\eta^2} && \Longrightarrow \frac{\dot{\eta}^2\left(\xi^2-\eta^2\right)}{a^2-\eta^2} = \frac{p_\eta^2\left(a^2-\eta^2\right)}{m^2\left(\xi^2-\eta^2\right)} \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \frac{m\dot{\phi}\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)}{a^2} && \Longrightarrow \frac{\dot{\phi}^2\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)}{a^2} = \frac{p_\phi^2 a^2}{m^2\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)}
\end{aligned}
$$

$$
H(\xi,\eta,\phi,p_\xi,p_\eta,p_\phi) = \frac{p_\xi^2\left(\xi^2-a^2\right)}{m^2\left(\xi^2-\eta^2\right)} + \frac{p_\eta^2\left(a^2-\eta^2\right)}{m^2\left(\xi^2-\eta^2\right)} + \frac{p_\phi^2 a^2}{m^2\left(\xi^2-a^2\right)\left(a^2-\eta^2\right)} - \frac{A}{\xi+\eta} - \frac{B}{\xi-\eta}
$$

```
H = D[L, ξ′[t]] ξ′[t] + D[L, η′[t]] η′[t] + D[L, ϕ′[t]] ϕ′[t] - L // FullSimplify
```

$$
\frac{1}{4}\left(-\frac{4\,B}{\sqrt{\left(\eta[t]-\xi[t]\right)^2}} - \frac{4\,A}{\sqrt{\left(\eta[t]+\xi[t]\right)^2}} + \frac{2\,m\left(-\eta[t]^2+\xi[t]^2\right)\eta'[t]^2}{a^2-\eta[t]^2} + \frac{2\left(a^2\,m\left(\eta[t]^2-\xi[t]^2\right)\xi'[t]^2 - m\left(a^2-\eta[t]^2\right)\left(a^2-\xi[t]^2\right)^2\phi'[t]^2\right)}{a^4-a^2\,\xi[t]^2}\right)
$$

```
K = -(m (η[t]^2 - ξ[t]^2) ((a^2 - ξ[t]^2) η'[t]^2 + (-a^2 + η[t]^2) ξ'[t]^2))/(2 (a^2 - η[t]^2) (a^2 - ξ[t]^2)) -
    (m (a^2 - η[t]^2) (a^2 - ξ[t]^2) ϕ'[t]^2)/(2 a^2);
U = -(B/Sqrt[(η[t] - ξ[t])^2] + A/Sqrt[(η[t] + ξ[t])^2]);
L - K + U // FullSimplify
H - K - U // FullSimplify
```

0

0

```
D[L, ξ'[t]] // FullSimplify
D[L, η'[t]] // FullSimplify
D[L, ϕ'[t]] // FullSimplify
```

$$\frac{m\left(\eta[t]^2-\xi[t]^2\right)\xi'[t]}{a^2-\xi[t]^2}$$

$$\frac{m\left(-\eta[t]^2+\xi[t]^2\right)\eta'[t]}{a^2-\eta[t]^2}$$

$$\frac{m\left(a^2-\eta[t]^2\right)\left(-a^2+\xi[t]^2\right)\phi'[t]}{a^2}$$