MAU23401: Advanced Classical Mechanics II
Homework 4 due 26/02/2021

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I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at
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Problem 1.

(i)

$$\begin{aligned}\{P,Q\} &= \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} \\ &= \beta - \delta + (\alpha-\gamma)\,e^{(q+p)^2} \\ &= 1\end{aligned}$$

$$\begin{aligned}(\alpha-\gamma)\,e^{(p+q)^2} &= 0 \\ \implies \alpha &= \gamma\end{aligned} \qquad \begin{aligned}\beta-\delta &= 1 \\ \implies \beta &= \delta+1\end{aligned}$$

```
P[q_, p_] = p + q;
Q[q_, p_] = q (α E^(q+p)^2 + β) + p (γ E^(q+p)^2 + δ);
D[P[q, p], p] × D[Q[q, p], q] - D[P[q, p], q] × D[Q[q, p], p] // FullSimplify
β + ⅇ^(p+q)^2 (α - γ) - δ
```

(ii)

$$\begin{aligned}\{P,Q\} &= \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} \\ &= q^{\alpha-1}p^{\beta-1}(\beta\delta-\alpha\gamma) \\ &= 1 \\ q^{\alpha-1}p^{\beta-1} &= 1 \\ \implies \alpha &= \beta = 1 \\ \beta\delta - \alpha\gamma &= 1 \\ \implies \delta &= \gamma+1\end{aligned}$$

```
P[q_, p_] = q^α p^β;
Q[q_, p_] = Log[p^γ q^δ];
D[P[q, p], p] × D[Q[q, p], q] - D[P[q, p], q] × D[Q[q, p], p] // FullSimplify
p^(-1+β) q^(-1+α) (-α γ + β δ)
```

Problem 2.

(a)

$$
\begin{aligned}
\{L_1, L_2\} &= \frac{\partial L_1}{\partial p_j}\frac{\partial L_2}{\partial q_j} - \frac{\partial L_1}{\partial q_j}\frac{\partial L_2}{\partial p_j} \\
&= \frac{\partial}{\partial p_j}(p_i q_i)\frac{\partial}{\partial q_j}\left(\frac{1}{2}(p_i p_i - q_i q_i)\right) - \frac{\partial}{\partial q_j}(p_i q_i)\frac{\partial}{\partial p_j}\left(\frac{1}{2}(p_i p_i - q_i q_i)\right) \\
&= \frac{1}{2}\left(q_j(-2q_j) - p_j(-2p_j)\right) \\
&= -\left(p_j p_j + q_j q_j\right) \\
\{L_1, L_2\} &= -2L_3
\end{aligned}
$$

$$
\begin{aligned}
\{L_1, L_3\} &= \frac{\partial L_1}{\partial p_j}\frac{\partial L_3}{\partial q_j} - \frac{\partial L_1}{\partial q_j}\frac{\partial L_3}{\partial p_j} \\
&= \frac{\partial}{\partial p_j}(p_i q_i)\frac{\partial}{\partial q_j}\left(\frac{1}{2}(p_i p_i + q_i q_i)\right) - \frac{\partial}{\partial q_j}(p_i q_i)\frac{\partial}{\partial p_j}\left(\frac{1}{2}(p_i p_i + q_i q_i)\right) \\
&= \frac{1}{2}\left(q_j(2q_j) - p_j(2p_j)\right) \\
&= -\left(p_j p_j - q_j q_j\right) \\
\{L_1, L_3\} &= -2L_2
\end{aligned}
$$

$$
\begin{aligned}
\{L_2, L_3\} &= \frac{\partial L_2}{\partial p_j}\frac{\partial L_3}{\partial q_j} - \frac{\partial L_2}{\partial q_j}\frac{\partial L_3}{\partial p_j} \\
&= \frac{\partial}{\partial p_j}\left(\frac{1}{2}(p_i p_i - q_i q_i)\right)\frac{\partial}{\partial q_j}\left(\frac{1}{2}(p_i p_i + q_i q_i)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}(p_i p_i - q_i q_i)\right)\frac{\partial}{\partial p_j}\left(\frac{1}{2}(p_i p_i + q_i q_i)\right) \\
&= \frac{1}{4}\left(2p_j(2q_j) + 2q_j(2p_j)\right) \\
&= 2p_j q_j \\
\{L_2, L_3\} &= 2L_1
\end{aligned}
$$

(b)

$$
f(q,p) \to f(Q,P) = f(q,p) + \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{\{V,\{V,\cdots\{}_{n}V,f\}\cdots\}
$$

$$
\Longrightarrow \quad q_i \to Q_i = q_j + \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{\{V,\{V,\cdots\{}_{n}V,q_i\}\cdots\}
$$

$$
p_i \to P_i = p_i + \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{\{V,\{V,\cdots\{}_{n}V,p_i\}\cdots\}
$$

(i)

$$
\begin{aligned}
\{V, q_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial q_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial q_j}{\partial p_k} \\
&= \frac{\partial}{\partial p_j}(\lambda p_i q_i) - 0 \\
&= \lambda q_j \\
\{V, \{V, q_j\}\} &= \{V, \lambda q_j\} \\
&= \lambda \{V, q_j\} \\
&= \lambda^2 q_j \\
\{V, \{V, \{V, q_j\}\}\} &= \{V, \lambda^2 q_j\} \\
&= \lambda^3 q_j \\
\underbrace{\{V, \{V, \ldots \{}_{n} V, q_j\} \ldots\} &= \lambda^n q_j
\end{aligned}
$$

$$
\begin{aligned}
Q_j &= q_j + \sum_{n=1}^{\infty} \frac{\lambda^n q_j}{n!} \\
&= q_j \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&= q_j\, e^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\{V, p_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial p_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial p_j}{\partial p_k} \\
&= 0 - \frac{\partial}{\partial q_j}(\lambda p_i q_i) \\
&= -\lambda p_j \\
\{V, \{V, p_j\}\} &= \{V, -\lambda p_j\} \\
&= -\lambda \{V, p_j\} \\
&= \lambda^2 p_j \\
\{V, \{V, \{V, p_j\}\}\} &= \{V, \lambda^2 p_j\} \\
&= -\lambda^3 p_j \\
\underbrace{\{V, \{V, \ldots \{}_{n} V, p_j\} \ldots\} &= (-\lambda)^n p_j
\end{aligned}
$$

$$
\begin{aligned}
P_j &= p_j + \sum_{n=1}^{\infty} \frac{(-\lambda)^n p_j}{n!} \\
&= p_j \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \\
&= p_j\, e^{-\lambda}
\end{aligned}
$$

$$
Q(q,p) = q\, e^{\lambda} \qquad\qquad P(q,p) = p\, e^{-\lambda}
$$

(ii)

$$\begin{aligned}
\{V, q_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial q_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial q_j}{\partial p_k} \\
&= \frac{\partial}{\partial p_j}\left(\frac{\lambda}{2}\left(p_i p_i - q_i q_i\right)\right) - 0 \\
&= \frac{\lambda}{2}\left(2p_j\right) \\
&= \lambda p_j \\
\{V, p_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial p_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial p_j}{\partial p_k} \\
&= 0 - \frac{\partial}{\partial q_j}\left(\frac{\lambda}{2}\left(p_i p_i - q_i q_i\right)\right) \\
&= -\frac{\lambda}{2}\left(-2q_j\right) \\
&= \lambda q_j
\end{aligned}$$

$$\begin{aligned}
\{V, \{V, q_j\}\} &= \{V, \lambda p_j\} \\
&= \lambda\{V, p_j\} \\
&= \lambda^2 q_j \\
\{V, \{V, \{V, q_j\}\}\} &= \{V, \lambda^2 q_j\} \\
&= \lambda^3 p_j \\
\{V, \{V, \{V, \{V, q_j\}\}\}\} &= \{V, \lambda^3 p_j\} \\
&= \lambda^4 q_j \\
\underbrace{\{V, \{V, \dots \{}_{n} V, q_j\} \dots\} &= \begin{cases} \lambda^n q_j & \text{if } n \text{ is even} \\ \lambda^n p_j & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
Q_j &= q_j + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\lambda^n q_j}{n!} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\lambda^n p_j}{n!} \\
&= q_j \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\lambda^n}{n!} + p_j \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\lambda^n}{n!} \\
&= q_j \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} + p_j \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \\
&= q_j \cosh\lambda + p_j \sinh\lambda
\end{aligned}$$

$$
\begin{aligned}
\{V,\{V,p_j\}\} &= \{V,\lambda q_j\} \\
&= \lambda\{V,q_j\} \\
&= \lambda^2 p_j \\
\{V,\{V,\{V,p_j\}\}\} &= \{V,\lambda^2 p_j\} \\
&= \lambda^3 q_j \\
\{V,\{V,\{V,\{V,p_j\}\}\}\} &= \{V,\lambda^3 q_j\} \\
&= \lambda^4 p_j \\
\underbrace{\{V,\{V,\ldots\{}_{n}V,p_j\}\ldots\} &= \begin{cases} \lambda^n q_j & \text{if } n \text{ is odd} \\ \lambda^n p_j & \text{if } n \text{ is even} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P_j &= p_j + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\lambda^n q_j}{n!} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\lambda^n p_j}{n!} \\
&= q_j \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\lambda^n}{n!} + p_j \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{\lambda^n}{n!} \\
&= q_j \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} + p_j \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \\
&= q_j \sinh\lambda + p_j \cosh\lambda
\end{aligned}
$$

$Q(q,p) = q\cosh\lambda + p\sinh\lambda$ $\qquad$ $P(q,p) = q\sinh\lambda + p\cosh\lambda$

(iii)

$$
\begin{aligned}
\{V,q_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial q_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial q_j}{\partial p_k} \\
&= \frac{\partial}{\partial p_j}\left(\frac{\lambda}{2}\left(p_i p_i + q_i q_i\right)\right) - 0 \\
&= \frac{\lambda}{2}\left(2p_j\right) \\
&= \lambda p_j \\
\{V,p_j\} &= \frac{\partial V}{\partial p_k}\frac{\partial p_j}{\partial q_k} - \frac{\partial V}{\partial q_k}\frac{\partial p_j}{\partial p_k} \\
&= 0 - \frac{\partial}{\partial q_j}\left(\frac{\lambda}{2}\left(p_i p_i + q_i q_i\right)\right) \\
&= -\frac{\lambda}{2}\left(2q_j\right) \\
&= -\lambda q_j
\end{aligned}
$$

$$
\begin{aligned}
\{V,\{V,q_j\}\} &= \{V,\lambda p_j\} \\
&= \lambda\{V,p_j\} \\
&= -\lambda^2 q_j \\
\{V,\{V,\{V,q_j\}\}\} &= \{V,-\lambda^2 q_j\} \\
&= -\lambda^3 p_j \\
\{V,\{V,\{V,\{V,q_j\}\}\}\} &= \{V,-\lambda^3 p_j\} \\
&= \lambda^4 q_j \\
\{V,\{V,\{V,\{V,\{V,q_j\}\}\}\}\} &= \{V,\lambda^4 q_j\} \\
&= \lambda^5 p_j \\
Q_j &= q_j + \frac{\lambda p_j}{1!} - \frac{\lambda^2 q_j}{2!} - \frac{\lambda^3 p_j}{3!} + \frac{\lambda^4 q_j}{4!} + \frac{\lambda^5 p_j}{5!} - \cdots \\
&= q_j\left(1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \cdots\right) + p_j\left(\frac{\lambda}{1!} - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} - \cdots\right) \\
&= q_j \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n)!} + p_j \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1}}{(2n+1)!} \\
&= q_j \cos\lambda + p_j \sin\lambda
\end{aligned}
$$

$$
\begin{aligned}
\{V,\{V,p_j\}\} &= \{V,-\lambda q_j\} \\
&= -\lambda\{V,q_j\} \\
&= -\lambda^2 p_j \\
\{V,\{V,\{V,p_j\}\}\} &= \{V,-\lambda^2 p_j\} \\
&= \lambda^3 q_j \\
\{V,\{V,\{V,\{V,p_j\}\}\}\} &= \{V,\lambda^3 q_j\} \\
&= \lambda^4 p_j \\
\{V,\{V,\{V,\{V,\{V,p_j\}\}\}\}\} &= \{V,\lambda^4 p_j\} \\
&= -\lambda^5 q_j \\
P_j &= p_j - \frac{\lambda q_j}{1!} - \frac{\lambda^2 p_j}{2!} + \frac{\lambda^3 q_j}{3!} + \frac{\lambda^4 p_j}{4!} - \frac{\lambda^5 q_j}{5!} - \cdots \\
&= q_j\left(-\frac{\lambda}{1!} + \frac{\lambda^3}{3!} - \frac{\lambda^5}{5!} + \cdots\right) + p_j\left(1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \cdots\right) \\
&= -q_j \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1}}{(2n+1)!} + p_j \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n}}{(2n)!} \\
&= -q_j \sin\lambda + p_j \cos\lambda
\end{aligned}
$$

$Q(q,p) = q\cos\lambda + p\sin\lambda$ $\qquad$ $P(q,p) = -q\sin\lambda + p\cos\lambda$