MAU23401: Advanced Classical Mechanics II
Homework 3 due 19/02/2021

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Problem 1.

(i)

$$\begin{aligned} p &= \frac{\partial F_1}{\partial q} \\ &= -\frac{m}{2}\frac{2(Q-q)(-1)}{t_2-t_1} \\ &= \frac{m(Q-q)}{t_2-t_1} \end{aligned}$$

$$\begin{aligned} P &= -\frac{\partial F_1}{\partial Q} \\ &= \frac{m}{2}\frac{2(Q-q)}{t_2-t_1} \\ &= \frac{m(Q-q)}{t_2-t_1} \end{aligned}$$

$$\begin{aligned} Q(q,p) &= q + \frac{p(t_2-t_1)}{m} \\ P(q,p) &= p \end{aligned}$$

The position is shifted by a constant amount of the momentum, yet the momentum remains unchanged.

(ii)

$$\begin{aligned} p_n &= \frac{\partial F_2}{\partial q^n} \\ &= \delta_n^i P_i + \epsilon_{ijk} a_i \delta_n^j P_k \\ &= P_n + \epsilon_{ink} a_i P_k \\ p_n - P_n &= \epsilon_{ink} a_i P_k \\ (p_n - P_n)^2 &= (\epsilon_{ink} a_i P_k)^2 \qquad \text{(fixed } n\text{, sum over } i,j,k) \\ &= 0 \end{aligned}$$

$$\begin{aligned} Q^n &= \frac{\partial F_2}{\partial P_n} \\ &= q^i \delta_n^i + \epsilon_{ijk} a_i q^j \delta_n^k \\ &= q^n + \epsilon_{ijn} a_i q^j \end{aligned}$$

$$\begin{aligned} Q^n(q,p) &= q^n + \epsilon_{ijn} a_i q^j \\ P_n(q,p) &= p_n \end{aligned}$$

The positions are infinitesimally shifted by a factor of the original position, yet the momenta remain unchanged.

(iii)

$$
\begin{aligned}
q^n &= -\frac{\partial F_3}{\partial p_n} \\
&= \delta^i_n Q^i - a^i \delta^i_n \\
&= Q^n - a^n
\end{aligned}
\qquad\qquad
\begin{aligned}
P_n &= -\frac{\partial F_3}{\partial Q^n} \\
&= p_i \delta^i_n - 0 \\
&= p_n
\end{aligned}
$$

$$
\begin{aligned}
Q^n(q,p) &= q^n + a^n \\
P_n(q,p) &= p_n
\end{aligned}
$$

The positions are shifted by the corresponding component of the vector a, yet the momenta remain unchanged.

(iv)

$$
\begin{aligned}
q^n &= -\frac{\partial F_3}{\partial p_n} \\
&= \delta^i_n Q^i + \tau \frac{\partial H}{\partial p_n} \\
&= Q^n + \tau \dot{Q}^n \\
(q^n - Q^n)^2 &= \tau^2 \left(\dot{Q}^n\right)^2 \qquad \text{(fixed } n\text{)} \\
&= 0
\end{aligned}
\qquad\qquad
\begin{aligned}
P_n &= -\frac{\partial F_3}{\partial Q^n} \\
&= p_i \delta^i_n + \tau \frac{\partial H}{\partial Q^n} \\
&= p_n + \tau \dot{p}_n \\
(P_n - p_n)^2 &= \tau^2 \left(\dot{p}_n\right)^2 \qquad \text{(fixed } n\text{)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Q^n(q,p) &= q^n \\
P_n(q,p) &= p_n
\end{aligned}
$$

Both positions and momenta remain unchanged.

Problem 2.

(i)

$$\begin{aligned}
\frac{p}{q} &= \sqrt{\frac{2P}{m\omega}}\sin Q\frac{1}{\sqrt{2m\omega P}\cos Q} \\
p &= m\omega q\cot Q \\
&= m\omega q\cot\left(\arcsin\left(q\sqrt{\frac{m\omega}{2P}}\right)\right) \\
&= \frac{\partial F_2}{\partial q} \\
\implies F_2 &= \int m\omega q\cot\left(\arcsin\left(q\sqrt{\frac{m\omega}{2P}}\right)\right)dq \\
&= P\sqrt{\frac{m\omega}{P}}\left(\frac{q}{2}\sqrt{2-\frac{mq^2\omega}{P}}+\sqrt{\frac{P}{m\omega}}\arcsin\left(q\sqrt{\frac{m\omega}{2P}}\right)\right) \\
F_2(q,P) &= \frac{q}{2}\sqrt{m\omega\left(2P-m\omega q^2\right)}+P\arcsin\left(q\sqrt{\frac{m\omega}{2P}}\right)
\end{aligned}$$

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Integrate[m ω q Cot[ArcSin[q Sqrt[m ω/(2 P)]]], q] // FullSimplify
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$$P\sqrt{\frac{m\,\omega}{P}}\left(\frac{1}{2}q\sqrt{2-\frac{m\,q^2\,\omega}{P}}+\frac{\sqrt{P}\,\text{ArcSin}\left[\frac{\sqrt{m}\,q\,\sqrt{\omega}}{\sqrt{2}\,\sqrt{P}}\right]}{\sqrt{m}\,\sqrt{\omega}}\right)$$

(ii)

$$\begin{aligned}
H(q,p) &= K+U \\
&= \frac{p^2}{2m}+\frac{m\omega^2q^2}{2} \\
\implies H(Q,P) &= \frac{1}{2m}2m\omega P\cos^2 Q+\frac{m\omega^2}{2}\frac{2P}{m\omega}\sin^2 Q \\
&= \omega P\cos^2 Q+\omega P\sin^2 Q \\
H(Q,P) &= \omega P
\end{aligned}$$

Since $\dot{P}=-\frac{\partial H}{\partial Q}=0$ and $\dot{Q}=\frac{\partial H}{\partial P}=\omega$, the motion of the harmonic oscillator in terms of Q and P is simply circular motion of angular velocity ω.

Problem 3.

(i)

$$
\begin{aligned}
P_i - p_i &= \frac{\partial \tilde{L}}{\partial \dot{q}^i} - \frac{\partial L}{\partial \dot{q}^i} \\
&= \frac{\partial L}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{q}^i}\frac{df}{dt} - \frac{\partial L}{\partial \dot{q}^i} \\
&= \frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial f}{\partial q^j}\dot{q}^j + \frac{\partial f}{\partial t}\right) \\
&= \frac{\partial^2 f}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial f}{\partial q^j}\delta^i_j + \frac{\partial^2 f}{\partial \dot{q}^i \partial t} \\
P_i - p_i &= \frac{\partial f}{\partial q^i}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H} - H &= P_i \dot{Q}^i - \tilde{L} - p_i q^i + L \\
&= P_i \dot{q}^i - L - \frac{df}{dt} - p_i q^i + L \\
&= (P_i - p_i)\,\dot{q}^i - \frac{\partial f}{\partial q^j}\dot{q}^j - \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial q^i}\dot{q}^i - \frac{\partial f}{\partial q^j}\dot{q}^j - \frac{\partial f}{\partial t} \\
\tilde{H} - H &= -\frac{\partial f}{\partial t}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\left\{Q^i, Q^j\right\} &= \left\{q^i, q^j\right\} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\{P_i, P_j\} &= \frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial q^k} - \frac{\partial P_i}{\partial q^k}\frac{\partial P_j}{\partial p_k} \\
&= \frac{\partial}{\partial p_k}\left(p_i + \frac{\partial f}{\partial q^i}\right)\frac{\partial}{\partial q^k}\left(p_j + \frac{\partial f}{\partial q^j}\right) - \frac{\partial}{\partial q^k}\left(p_i + \frac{\partial f}{\partial q^i}\right)\frac{\partial}{\partial p_k}\left(p_j + \frac{\partial f}{\partial q^j}\right) \\
&= \left(\delta^i_k + 0\right)\left(0 + \frac{\partial^2 f}{\partial q^k \partial q^j}\right) - \left(0 + \frac{\partial^2 f}{\partial q^k \partial q^i}\right)\left(\delta^j_k + 0\right) \\
&= \frac{\partial^2 f}{\partial q^i \partial q^j} - \frac{\partial^2 f}{\partial q^j \partial q^i} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\{P_i, Q^j\right\} &= \left\{P_i, q^j\right\} \\
&= \frac{\partial P_i}{\partial p_k}\frac{\partial q^j}{\partial q^k} - \frac{\partial P_i}{\partial q^k}\frac{\partial q^j}{\partial p_k} \\
&= \frac{\partial}{\partial p_k}\left(p_i + \frac{\partial f}{\partial q^i}\right)\delta^j_k - \frac{\partial P_i}{\partial q^k}(0) \\
&= \delta^j_k\left(\delta^i_k + 0\right) - 0 \\
&= \delta^i_j
\end{aligned}
$$

(iii)

$$\begin{aligned} p_i &= P_i - \frac{\partial f}{\partial q^i} \\ &= \frac{\partial F_2}{\partial q^i} \\ \implies F_2 &= \int \left(P_i - \frac{\partial f}{\partial q^i} \right) dq^i \\ F_2(q, P, t) &= P_i q^i - f(q, t) \end{aligned}$$