MAU23402: Advanced Classical Mechanics II
Homework 2 due 12/02/2021

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Problem 1.

$$
\begin{aligned}
\{\alpha,\{\beta,\gamma\}\} &= \frac{\partial\alpha}{\partial p_i}\frac{\partial\{\beta,\gamma\}}{\partial q^i} - \frac{\partial\alpha}{\partial q^i}\frac{\partial\{\beta,\gamma\}}{\partial p_i} \\
&= \frac{\partial\alpha}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial\beta}{\partial p_j}\frac{\partial\gamma}{\partial q^j} - \frac{\partial\beta}{\partial q^j}\frac{\partial\gamma}{\partial p_j}\right) - \frac{\partial\alpha}{\partial q^i}\frac{\partial}{\partial p_i}\left(\frac{\partial\beta}{\partial p_j}\frac{\partial\gamma}{\partial q^j} - \frac{\partial\beta}{\partial q^j}\frac{\partial\gamma}{\partial p_j}\right) \\
&= \frac{\partial\alpha}{\partial p_i}\left(\frac{\partial^2\beta}{\partial q^i\partial p_j}\frac{\partial\gamma}{\partial q^j} + \frac{\partial\beta}{\partial p_j}\frac{\partial^2\gamma}{\partial q^i\partial q^j} - \frac{\partial^2\beta}{\partial q^i\partial q^j}\frac{\partial\gamma}{\partial p_j} - \frac{\partial\beta}{\partial q^j}\frac{\partial^2\gamma}{\partial q^i\partial p_j}\right) \\
&\quad - \frac{\partial\alpha}{\partial q^i}\left(\frac{\partial^2\beta}{\partial p_i\partial p_j}\frac{\partial\gamma}{\partial q^j} + \frac{\partial\beta}{\partial p_j}\frac{\partial^2\gamma}{\partial p_i\partial q^j} - \frac{\partial^2\beta}{\partial p_i\partial q^j}\frac{\partial\gamma}{\partial p_j} - \frac{\partial\beta}{\partial q^j}\frac{\partial^2\gamma}{\partial p_i\partial p_j}\right) \\
\implies & \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} \\
&= \frac{\partial f}{\partial p_i}\left(\frac{\partial^2 g}{\partial q^i\partial p_j}\frac{\partial h}{\partial q^j} + \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^i\partial q^j} - \frac{\partial^2 g}{\partial q^i\partial q^j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial q^i\partial p_j}\right) \\
&\quad - \frac{\partial f}{\partial q^i}\left(\frac{\partial^2 g}{\partial p_i\partial p_j}\frac{\partial h}{\partial q^j} + \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_i\partial q^j} - \frac{\partial^2 g}{\partial p_i\partial q^j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_i\partial p_j}\right) \\
&\quad + \frac{\partial g}{\partial p_i}\left(\frac{\partial^2 h}{\partial q^i\partial p_j}\frac{\partial f}{\partial q^j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^i\partial q^j} - \frac{\partial^2 h}{\partial q^i\partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial q^i\partial p_j}\right) \\
&\quad - \frac{\partial g}{\partial q^i}\left(\frac{\partial^2 h}{\partial p_i\partial p_j}\frac{\partial f}{\partial q^j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_i\partial q^j} - \frac{\partial^2 h}{\partial p_i\partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_i\partial p_j}\right) \\
&\quad + \frac{\partial h}{\partial p_i}\left(\frac{\partial^2 f}{\partial q^i\partial p_j}\frac{\partial g}{\partial q^j} + \frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q^i\partial q^j} - \frac{\partial^2 f}{\partial q^i\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial q^i\partial p_j}\right) \\
&\quad - \frac{\partial h}{\partial q^i}\left(\frac{\partial^2 f}{\partial p_i\partial p_j}\frac{\partial g}{\partial q^j} + \frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_i\partial q^j} - \frac{\partial^2 f}{\partial p_i\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial p_i\partial p_j}\right) \\
&= \frac{\partial f}{\partial p_i}\left(\frac{\partial^2 g}{\partial q^i\partial p_j}\frac{\partial h}{\partial q^j} + \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^i\partial q^j} - \frac{\partial^2 g}{\partial q^i\partial q^j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial q^i\partial p_j}\right. \\
&\quad \left. - \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^j\partial q^i} + \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_j\partial q^i} + \frac{\partial^2 g}{\partial q^j\partial q^i}\frac{\partial h}{\partial p_j} - \frac{\partial^2 g}{\partial p_j\partial q^i}\frac{\partial h}{\partial q^j}\right) \\
&\quad + \frac{\partial f}{\partial q^i}\left(-\frac{\partial^2 g}{\partial p_i\partial p_j}\frac{\partial h}{\partial q^j} - \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_i\partial q^j} + \frac{\partial^2 g}{\partial p_i\partial q^j}\frac{\partial h}{\partial p_j} + \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_i\partial p_j}\right. \\
&\quad \left. + \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^j\partial p_i} - \frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_j\partial p_i} - \frac{\partial^2 g}{\partial q^j\partial p_i}\frac{\partial h}{\partial p_j} + \frac{\partial^2 g}{\partial p_j\partial p_i}\frac{\partial h}{\partial q^j}\right) \\
&\quad + \frac{\partial g}{\partial p_i}\left(\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^i\partial q^j} - \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial q^i\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^j\partial q^i} + \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_j\partial q^i}\right) \\
&\quad - \frac{\partial g}{\partial q^i}\left(\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_i\partial q^j} - \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_i\partial p_j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^j\partial p_i} - \frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_j\partial p_i}\right) \\
&= \frac{\partial f}{\partial p_i}(0) + \frac{\partial f}{\partial q^i}(0) + \frac{\partial g}{\partial p_i}(0) + \frac{\partial g}{\partial q^i}(0) \text{ assuming } f,g,h \text{ are continuous} \\
&= 0
\end{aligned}
$$

Problem 2.

(a)

$$\begin{aligned}\{J_{ab}, p_c\} &= \frac{\partial J_{ab}}{\partial p_k}\frac{\partial p_c}{\partial x_k} - \frac{\partial J_{ab}}{\partial x_k}\frac{\partial p_c}{\partial p_k} \\ &= 0 - \delta_{ck}\frac{\partial}{\partial x_k}(p_a x_b - p_b x_a) \\ &= -\delta_{ck}(p_a\delta_{bk} - p_b\delta_{ak}) \\ \{J_{ab}, p_c\} &= \delta_{ac}p_b - \delta_{bc}p_a\end{aligned}$$

$$\begin{aligned}\{J_{ab}, x_c\} &= \frac{\partial J_{ab}}{\partial p_k}\frac{\partial x_c}{\partial x_k} - \frac{\partial J_{ab}}{\partial x_k}\frac{\partial x_c}{\partial p_k} \\ &= \delta_{ck}\frac{\partial}{\partial p_k}(p_a x_b - p_b x_a) - 0 \\ &= \delta_{ck}(x_b\delta_{ak} - x_a\delta_{bk}) \\ \{J_{ab}, x_c\} &= \delta_{ac}x_b - \delta_{bc}x_a\end{aligned}$$

$$\begin{aligned}\{J_{ab}, J_{cd}\} &= \frac{\partial J_{ab}}{\partial p_k}\frac{\partial J_{cd}}{\partial x_k} - \frac{\partial J_{ab}}{\partial x_k}\frac{\partial J_{cd}}{\partial p_k} \\ &= \frac{\partial}{\partial p_k}(p_a x_b - p_b x_a)\frac{\partial}{\partial x_k}(p_c x_d - p_d x_c) - \frac{\partial}{\partial x_k}(p_a x_b - p_b x_a)\frac{\partial}{\partial p_k}(p_c x_d - p_d x_c) \\ &= (x_b\delta_{ak} - x_a\delta_{bk})(p_c\delta_{dk} - p_d\delta_{ck}) - (p_a\delta_{bk} - p_b\delta_{ak})(x_d\delta_{ck} - x_c\delta_{dk}) \\ &= \delta_{ad}p_c x_b - \delta_{ac}p_d x_b - \delta_{bd}p_c x_a + \delta_{bc}p_d x_a - (\delta_{bc}p_a x_d - \delta_{bd}p_a x_c - \delta_{ac}p_b x_d + \delta_{ad}p_b x_c) \\ &= \delta_{ad}(p_c x_b - p_b x_c) + \delta_{ac}(p_b x_d - p_d x_b) + \delta_{bd}(p_a x_c - p_c x_a) + \delta_{bc}(p_d x_a - p_a x_d) \\ \{J_{ab}, J_{cd}\} &= \delta_{ad}J_{cb} + \delta_{ac}J_{bd} + \delta_{bd}J_{ac} + \delta_{bc}J_{da}\end{aligned}$$

$$\begin{aligned}\{J_{ab}, p^2\} = \{J_{ab}, p_i^2\} &= \frac{\partial J_{ab}}{\partial p_k}\frac{\partial p_i^2}{\partial x_k} - \frac{\partial J_{ab}}{\partial x_k}\frac{\partial p_i^2}{\partial p_k} \\ &= 0 - 2p_i\delta_{ik}\frac{\partial}{\partial x_k}(p_a x_b - p_b x_a) \\ &= -2p_i\delta_{ik}(p_a\delta_{bk} - p_b\delta_{ak}) \\ &= 2(\delta_{ai}p_b p_i - \delta_{bi}p_a p_i) \\ &= 2(p_b p_a - p_a p_b) \\ &= 0\end{aligned}$$

$$\begin{aligned}\{J_{ab}, r^2\} = \{J_{ab}, x_i^2\} &= \frac{\partial J_{ab}}{\partial p_k}\frac{\partial x_i^2}{\partial x_k} - \frac{\partial J_{ab}}{\partial x_k}\frac{\partial x_i^2}{\partial p_k} \\ &= 2x_i\delta_{ik}\frac{\partial}{\partial p_k}(p_a x_b - p_b x_a) - 0 \\ &= 2x_i\delta_{ik}(\delta_{ak}x_b - \delta_{bk}x_a) \\ &= 2(\delta_{ai}x_b x_i - \delta_{bi}x_a x_i) \\ &= 2(x_b x_a - x_a x_b) \\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}
\{J_{ab}, J_{cd}\} &= \delta_{ad}J_{cb} + \delta_{ac}J_{bd} + \delta_{bd}J_{ac} + \delta_{bc}J_{da} \\
\implies \{J_{14}, J_{13}\} &= \delta_{13}J_{14} + \delta_{11}J_{43} + \delta_{43}J_{11} + \delta_{41}J_{31} \\
&= 0 + J_{43} + 0 + 0 \\
\{J_{14}, J_{13}\} &= J_{43} = p_4x_3 - p_3x_4
\end{aligned}$$

Problem 3.

(a)

$$\begin{aligned}
\{M_i, p_j\} &= \frac{\partial M_i}{\partial p_k}\frac{\partial p_j}{\partial x_k} - \frac{\partial M_i}{\partial x_k}\frac{\partial p_j}{\partial p_k} \\
&= 0 - \delta_{jk}\frac{\partial}{\partial x_k}(\epsilon_{imn}x_mp_n) \\
&= -\delta_{jk}\epsilon_{imn}\delta_{mk}p_n \\
&= -\delta_{jm}\epsilon_{imn}p_n \\
\{M_i, p_j\} &= -\epsilon_{ijk}p_k
\end{aligned}$$

$$\begin{aligned}
\{M_i, x_j\} &= \frac{\partial M_i}{\partial p_k}\frac{\partial x_j}{\partial x_k} - \frac{\partial M_i}{\partial x_k}\frac{\partial x_j}{\partial p_k} \\
&= \delta_{jk}\frac{\partial}{\partial p_k}(\epsilon_{imn}x_mp_n) - 0 \\
&= \delta_{jk}\epsilon_{imn}x_m\delta_{nk} \\
&= \delta_{jn}\epsilon_{imn}x_m \\
&= \epsilon_{imj}x_m \\
\{M_i, x_j\} &= -\epsilon_{ijk}x_k
\end{aligned}$$

$$\begin{aligned}
\{M_i, M_j\} &= \frac{\partial M_i}{\partial p_k}\frac{\partial M_j}{\partial x_k} - \frac{\partial M_i}{\partial x_k}\frac{\partial M_j}{\partial p_k} \\
&= \frac{\partial}{\partial p_k}(\epsilon_{imn}x_mp_n)\frac{\partial}{\partial x_k}(\epsilon_{jlo}x_lp_o) \\
&= \epsilon_{imn}x_m\delta_{kn}\epsilon_{jlo}\delta_{kl}p_o - \epsilon_{imn}\delta_{km}p_n\epsilon_{jlo}x_l\delta_{ko} \\
&= \delta_{ln}\epsilon_{imn}\epsilon_{jlo}x_mp_o - \delta_{mo}\epsilon_{imn}\epsilon_{jlo}x_lp_n \\
&= \epsilon_{imq}\epsilon_{jqo}x_mp_o - \epsilon_{iqn}\epsilon_{jlq}x_lp_n \\
&= \epsilon_{imq}\epsilon_{jqo}x_mp_o - \epsilon_{iqm}\epsilon_{joq}x_op_m \\
&= \epsilon_{imq}\epsilon_{jqo}x_mp_o - \epsilon_{imq}\epsilon_{jqo}x_op_m \\
&= \epsilon_{imq}\epsilon_{jqo}(x_mp_o - x_op_m) \\
&= \epsilon_{qmi}\epsilon_{qjo}(x_mp_o - x_op_m) \\
&= (\delta_{mj}\delta_{io} - \delta_{mo}\delta_{ij})(x_mp_o - x_op_m) \\
&= \delta_{mj}\delta_{io}x_mp_o - \delta_{mj}\delta_{io}x_op_m \text{ if } i \neq j \\
&= x_jp_i - x_ip_j \\
&= \epsilon_{jik}M_k \\
\{M_i, M_j\} &= -\epsilon_{ijk}M_k
\end{aligned}$$

$$
\begin{aligned}
\frac{dM_i}{dt} &= \frac{\partial M_i}{\partial t} + \{H, M_i\} \\
&= 0 \text{ as } M_i \text{ is an integral of motion} \\
\implies \{M_i, H\} &= \frac{\partial M_i}{\partial t} \\
&= 0 \text{ as } M_i \text{ does not depend on time}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\{D_i, p_j\} &= \frac{\partial D_i}{\partial p_k}\frac{\partial p_j}{\partial x_k} - \frac{\partial D_i}{\partial x_k}\frac{\partial p_j}{\partial p_k} \\
&= 0 - \delta_{jk}\frac{\partial D_i}{\partial x_k} \\
&= -\frac{\partial}{\partial x_j}\left(\frac{1}{m}\epsilon_{imn}\epsilon_{lon}x_l p_m p_o + \frac{\alpha x_i}{\sqrt{x_a^2}}\right) \\
&= -\left(\frac{1}{m}\epsilon_{imn}\epsilon_{lon}\delta_{lj}p_m p_o + \frac{\alpha\delta_{ij}}{\sqrt{x_a^2}} + \alpha x_i \frac{\partial}{\partial x_j}\left(x_a^2\right)^{-\frac{1}{2}}\right) \\
&= -\left(-\frac{1}{m}\epsilon_{nim}\epsilon_{noj}p_m p_o + \frac{\alpha\delta_{ij}}{r} + \alpha x_i\left(-\frac{1}{2}\left(x_b^2\right)^{-\frac{3}{2}}\left(2x_a\delta_{aj}\right)\right)\right) \\
&= -\left(-\frac{1}{m}(\delta_{io}\delta_{mj} - \delta_{ij}\delta_{mo})p_m p_o + \frac{\alpha\delta_{ij}}{r} - \frac{\alpha x_i x_j}{r^3}\right) \\
\{D_i, p_j\} &= \frac{1}{m}\left(p_i p_j - p^2\delta_{ij}\right) - \frac{\alpha\delta_{ij}}{r} + \frac{\alpha x_i x_j}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
\{D_i, x_j\} &= \frac{\partial D_i}{\partial p_k}\frac{\partial x_j}{\partial x_k} - \frac{\partial D_i}{\partial x_k}\frac{\partial x_j}{\partial p_k} \\
&= \delta_{jk}\frac{\partial D_i}{\partial p_k} - 0 \\
&= \frac{\partial}{\partial p_j}\left(\frac{1}{m}\epsilon_{imn}\epsilon_{lon}x_l p_m p_o + \frac{\alpha x_i}{\sqrt{x_a^2}}\right) \\
&= \frac{1}{m}\epsilon_{imn}\epsilon_{lon}x_l(\delta_{mj}p_o + p_m\delta_{oj}) \\
&= \frac{1}{m}(\epsilon_{nij}\epsilon_{nlo}x_l p_o + \epsilon_{nim}\epsilon_{nlj}x_l p_m) \\
&= \frac{1}{m}((\delta_{il}\delta_{jo} - \delta_{io}\delta_{jl})x_l p_o + (\delta_{il}\delta_{mj} - \delta_{ij}\delta_{ml})x_l p_m) \\
&= \frac{1}{m}(x_i p_j - x_j p_i + x_i p_j - \delta_{ij}x_k p_k) \\
\{D_i, x_j\} &= \frac{1}{m}\left(2x_i p_j - \delta_{ij}\left(\vec{r}\cdot\vec{p}\right) - p_i x_j\right)
\end{aligned}
$$

$$
\begin{aligned}
\{D_i, M_j\} &= -\{\epsilon_{jkl} x_k p_l, D_i\} \\
&= -\epsilon_{jkl}\left(p_l \frac{\partial D_i}{\partial p_k} - x_k \frac{\partial D_i}{\partial x_l}\right) \\
&= -\epsilon_{jkl}\left(\frac{p_l}{m}(2x_i p_k - \delta_{ik}(\vec{r}\cdot\vec{p}) - p_i x_k) - x_k\left(\frac{1}{m}\left(p_i p_l - p^2\delta_{il}\right) - \frac{\alpha\delta_{il}}{r} + \frac{\alpha x_i x_l}{r^3}\right)\right) \\
&= -\epsilon_{jkl}\left(\frac{p_l}{m}(-\delta_{ik}(\vec{r}\cdot\vec{p}) - p_i x_k) - x_k\left(\frac{1}{m}\left(p_i p_l - p^2\delta_{il}\right) - \frac{\alpha\delta_{il}}{r}\right)\right) \\
&= -\epsilon_{ijk}\left(\frac{1}{m}\left(x_k p^2 - p_k(\vec{r}\cdot\vec{p}) - \frac{\alpha x_k}{r}\right)\right)
\end{aligned}
$$

$$\{D_i, M_j\} = -\epsilon_{ijk} D_k$$

$$
\begin{aligned}
\frac{dD_i}{dt} &= \frac{\partial D_i}{\partial t} + \{H, D_i\} \\
&= 0 \text{ as } D_i \text{ is an integral of motion} \\
\implies \{D_i, H\} &= \frac{\partial D_i}{\partial t} \\
&= 0 \text{ as } D_i \text{ does not depend on time}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\{D_i, D_j\} &= \frac{\partial D_i}{\partial p_k}\frac{\partial D_j}{\partial x_k} - \frac{\partial D_i}{\partial x_k}\frac{\partial D_j}{\partial p_k} \\
&= \left(\frac{1}{m}\left(2x_ip_k - \delta_{ik}\left(\vec{r}\cdot\vec{p}\right) - p_ix_k\right)\right)\left(\frac{1}{m}\left(p^2\delta_{jk} - p_jp_k\right) + \frac{\alpha\delta_{jk}}{r} - \frac{\alpha x_jx_k}{r^3}\right) \\
&\quad - \left(\frac{1}{m}\left(2x_jp_k - \delta_{jk}\left(\vec{r}\cdot\vec{p}\right) - p_jx_k\right)\right)\left(\frac{1}{m}\left(p^2\delta_{ik} - p_ip_k\right) + \frac{\alpha\delta_{ik}}{r} - \frac{\alpha x_ix_k}{r^3}\right) \\
&= \frac{1}{m}\left[\frac{1}{m}\left(2x_ip_kp^2\delta_{jk} - 2x_ip_kp_jp_k\right) + \frac{\alpha}{r}\left(2x_ip_k\delta_{jk} - \frac{2x_ip_kx_jx_k}{r^2}\right)\right. \\
&\quad + \frac{1}{m}\left(-\delta_{ik}(\vec{r}\cdot\vec{p})p^2\delta_{jk} + \delta_{ik}(\vec{r}\cdot\vec{p})p_jp_k\right) + \frac{\alpha}{r}\left(\delta_{ik}(\vec{r}\cdot\vec{p})\delta_{jk} + \frac{\delta_{ik}(\vec{r}\cdot\vec{p})x_jx_k}{r^2}\right) \\
&\quad + \frac{1}{m}\left(-p_ix_kp^2\delta_{jk} + p_ix_kp_jp_k\right) + \frac{\alpha}{r}\left(-p_ix_k\delta_{jk} + \frac{p_ix_kx_jx_k}{r^2}\right) \\
&\quad - \frac{1}{m}\left(2x_jp_kp^2\delta_{ik} - 2x_jp_kp_ip_k\right) - \frac{\alpha}{r}\left(2x_jp_k\delta_{ik} - \frac{2x_jp_kx_ix_k}{r^2}\right) \\
&\quad - \frac{1}{m}\left(-\delta_{jk}(\vec{r}\cdot\vec{p})p^2\delta_{ik} + \delta_{jk}(\vec{r}\cdot\vec{p})p_ip_k\right) - \frac{\alpha}{r}\left(\delta_{jk}(\vec{r}\cdot\vec{p})\delta_{ik} + \frac{\delta_{jk}(\vec{r}\cdot\vec{p})x_ix_k}{r^2}\right) \\
&\quad \left. - \frac{1}{m}\left(-p_jx_kp^2\delta_{ik} + p_jx_kp_ip_k\right) - \frac{\alpha}{r}\left(-p_jx_k\delta_{ik} + \frac{p_jx_kx_ix_k}{r^2}\right)\right] \\
&= \frac{1}{m}\left[\frac{1}{m}\left(2x_ip_jp^2 - 2x_ip_jp^2 - \delta_{ij}(\vec{r}\cdot\vec{p})p^2 + (\vec{r}\cdot\vec{p})p_ip_j - x_jp_ip^2 + p_ip_j(\vec{r}\cdot\vec{p})\right.\right. \\
&\quad \left. - 2x_jp_ip^2 + 2x_jp_ip^2 + \delta_{ij}(\vec{r}\cdot\vec{p})p^2 - (\vec{r}\cdot\vec{p})p_ip_j + x_ip_jp^2 - p_ip_j(\vec{r}\cdot\vec{p})\right) \\
&\quad + \frac{\alpha}{r}\left(2x_ip_j - \frac{2x_ix_j(\vec{r}\cdot\vec{p})}{r^2} + \delta_{ij}(\vec{r}\cdot\vec{p}) + \frac{x_ix_j(\vec{r}\cdot\vec{p})}{r^2} - x_jp_i + \frac{x_jp_ir^2}{r^2}\right. \\
&\quad \left.\left. - 2x_jp_i + \frac{2x_ix_j(\vec{r}\cdot\vec{p})}{r^2} - \delta_{ij}(\vec{r}\cdot\vec{p}) - \frac{x_ix_j(\vec{r}\cdot\vec{p})}{r^2} + x_ip_j - \frac{x_ip_jr^2}{r^2}\right)\right] \\
&= \frac{1}{m}\left[\frac{1}{m}\left(p^2(x_ip_j - x_jp_i)\right) + \frac{\alpha}{r}\left(2(x_ip_j - x_jp_i)\right)\right] \\
&= \frac{2}{m}\left(\frac{p^2}{2m} + \frac{\alpha}{r}\right)(x_ip_j - x_jp_i)
\end{aligned}
$$

$$\{D_i, D_j\} = \frac{2H\epsilon_{ijk}M_k}{m}$$

(d)

$$\begin{aligned}
\{J_{ab}, J_{cd}\} &= \{-\epsilon_{abk}M_k, -\epsilon_{cdl}M_l\} \text{ if } a,b,c,d = 1,2,3 \\
&= \epsilon_{abk}\epsilon_{cdl}\{\epsilon_{kij}x_i p_j, \epsilon_{lmn}x_m p_n\} \\
&= \epsilon_{abk}\epsilon_{cdl}\epsilon_{kij}\epsilon_{lmn}\left(\frac{\partial x_i p_j}{\partial p_o}\frac{\partial x_m p_n}{\partial x_o} - \frac{\partial x_i p_j}{\partial x_o}\frac{\partial x_m p_n}{\partial p_o}\right) \\
&= \epsilon_{abk}\epsilon_{cdl}\epsilon_{kij}\epsilon_{lmn}\left(x_i\delta_{jo}\delta_{mo}p_n - \delta_{io}p_j x_m\delta_{no}\right) \\
&= \epsilon_{kab}\epsilon_{kij}\epsilon_{lcd}\epsilon_{lmn}\left(\delta_{mj}x_i p_n - \delta_{in}x_m p_j\right) \\
&= \epsilon_{kab}\epsilon_{kij}\epsilon_{lcd}\epsilon_{ljn}x_i p_n - \epsilon_{kab}\epsilon_{kij}\epsilon_{lcd}\epsilon_{lmi}x_m p_j \\
&= (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{dn} - \delta_{cn}\delta_{dj})x_i p_n - (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cm}\delta_{di} - \delta_{ci}\delta_{dm})x_m p_j \\
&= (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}x_i p_d - \delta_{dj}x_i p_c + \delta_{ci}x_d p_j - \delta_{di}x_c p_j) \\
&= \delta_{bc}x_a p_d - \delta_{bd}x_a p_c + \delta_{ac}x_d p_b - \delta_{ad}x_c p_b - \delta_{ac}x_b p_d + \delta_{ad}x_b p_c - \delta_{bc}x_d p_a + \delta_{bd}x_c p_a \\
&= \delta_{bc}(x_a p_d - x_d p_a) - \delta_{bd}(x_a p_c - x_c p_a) + \delta_{ac}(x_d p_b - x_b p_d) - \delta_{ad}(x_c p_b - x_b p - c) \\
&= \delta_{bc}J_{da} - \delta_{bd}J_{ca} + \delta_{ac}J_{bd} - \delta_{ad}J_{bc} \\
&= \eta_{ac}J_{bd} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac} \text{ for } a,b,c,d = 1,2,3
\end{aligned}$$