MAU23401: Advanced Classical Mechanics II
Homework 1 due 05/02/2021

Ruaidhrí Campion
19333850
SF Theoretical Physics

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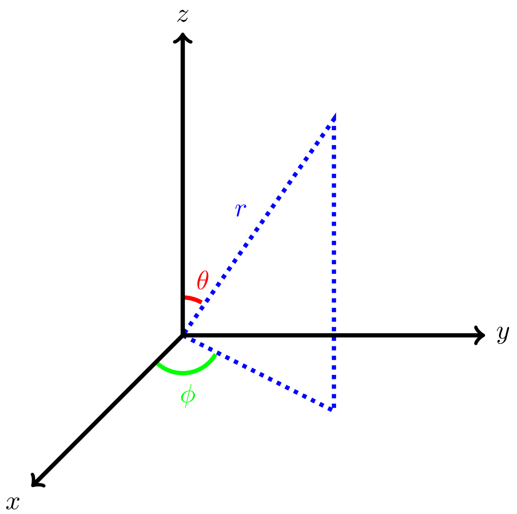
Problem 1.

(a)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$



$$\begin{aligned} \dot{\vec{r}}^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta \\ \implies L &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - U(r) \end{aligned}$$

```
x[t_] = r[t] Sin[θ[t]] Cos[ϕ[t]];
y[t_] = r[t] Sin[θ[t]] Sin[ϕ[t]];
z[t_] = r[t] Cos[θ[t]];
kineticL = m/2 ((D[x[t], t])^2 + (D[y[t], t])^2 + (D[z[t], t])^2) // FullSimplify
```

$$\frac{1}{2} m \left(r'[t]^2 + r[t]^2 \left(\theta'[t]^2 + \text{Sin}[\theta[t]]^2 \phi'[t]^2\right)\right)$$

(b)

$$
\begin{aligned}
H &= p_i \dot{q}^i - L \\
&= \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \\
&= m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) + U(r) \\
&= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) + U(r) \\
&= \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\theta}^2}{2} + \frac{mr^2\dot{\phi}^2\sin^2\theta}{2} + U(r)
\end{aligned}
$$

$$
\begin{aligned}
p_r &= m\dot{r} \\
\implies p_r^2 &= m^2\dot{r}^2 \\
\implies \frac{m\dot{r}^2}{2} &= \frac{p_r^2}{2m}
\end{aligned}
\qquad
\begin{aligned}
p_\theta &= mr^2\dot{\theta} \\
\implies p_\theta^2 &= m^2r^4\dot{\theta}^2 \\
\implies \frac{mr^2\dot{\theta}^2}{2} &= \frac{p_\theta^2}{2mr^2}
\end{aligned}
\qquad
\begin{aligned}
p_\phi &= mr^2\dot{\phi}\sin^2\theta \\
\implies p_\phi^2 &= m^2r^4\dot{\phi}\sin^4\theta \\
\implies \frac{mr^2\dot{\phi}^2\sin^2\theta}{2} &= \frac{p_\phi^2}{2mr^2\sin^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
\implies H &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + U(r) \\
H &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2\csc^2\theta}{r^2}\right) + U(r)
\end{aligned}
$$

```
D[kineticL, r'[t]] r'[t] + D[kineticL, ϕ'[t]] ϕ'[t] + D[kineticL, θ'[t]] θ'[t] //
  FullSimplify
```

```
m (r'[t]^2 + r[t]^2 (θ'[t]^2 + Sin[θ[t]]^2 ϕ'[t]^2))
```

(c)

$$
\begin{aligned}
\dot{p}_i &= -\frac{\partial H}{\partial q^i} \\
\dot{p}_r &= \frac{p_\theta^2 + p_\phi^2\csc^2\theta}{mr^3} \\
\dot{p}_\theta &= \frac{p_\phi^2\cot\theta\csc^2\theta}{mr^2} \\
\dot{p}_\phi &= 0
\end{aligned}
\qquad
\begin{aligned}
\dot{q}^i &= \frac{\partial H}{\partial p_i} \\
\dot{q}^r &= \frac{p_r}{m} \\
\dot{q}^\theta &= \frac{p_\theta}{mr^2} \\
\dot{q}^\phi &= \frac{p_\phi\csc^2\theta}{mr^2}
\end{aligned}
$$

```
H = 1/(2 m) (pr^2 + pθ^2/r^2 + pϕ^2/(r^2 (Sin[θ])^2));
-D[H, r] // FullSimplify
-D[H, θ] // FullSimplify
-D[H, ϕ] // FullSimplify
D[H, pr] // FullSimplify
D[H, pθ] // FullSimplify
D[H, pϕ] // FullSimplify
```

$$\frac{p\theta^2 + p\phi^2\,\mathrm{Csc}[\theta]^2}{m\,r^3}$$

$$\frac{p\phi^2\,\mathrm{Cot}[\theta]\,\mathrm{Csc}[\theta]^2}{m\,r^2}$$

$$0$$

$$\frac{pr}{m}$$

$$\frac{p\theta}{m\,r^2}$$

$$\frac{p\phi\,\mathrm{Csc}[\theta]^2}{m\,r^2}$$

Problem 2.

(a)

$$
\begin{aligned}
H &= \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L \\
&= m_1\dot{x}_1^2 + B_{12}\dot{x}_1 x_2 + m_2\dot{x}_2^2 + B_{21}x_1\dot{x}_2 - \frac{m_1}{2}\dot{x}_1^2 - \frac{m}{2}\dot{x}_2^2 - B_{12}\dot{x}_1 x_2 - B_{21}x_1\dot{x}_2 + U(x_1, x_2) \\
&= \frac{m_1\dot{x}_1^2}{2} + \frac{m_2\dot{x}_2^2}{2} + U(x_1, x_2)
\end{aligned}
$$

$$
\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{x}_1} \\
&= m_1\dot{x}_1 + B_{12}x_2 \\
\implies m_1^2\dot{x}_1^2 &= (p_1 - B_{12}x_2)^2 \\
\implies \frac{m_1\dot{x}_1^2}{2} &= \frac{(p_1 - B_{12}x_2)^2}{2m_1} \\
\text{Similarly, } \frac{m_2\dot{x}_2^2}{2} &= \frac{(p_2 - B_{21}x_1)^2}{2m_2}
\end{aligned}
$$

$$
\implies H = \frac{(p_1 - B_{12}x_2)^2}{2m_1} + \frac{(p_2 - B_{21}x_1)^2}{2m_2} + U(x_1, x_2)
$$

(b)

$$
\dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad \dot{x}_i = \frac{\partial H}{\partial p_i}
$$

$$
\dot{p}_1 = \frac{B_{21}(p_2 - B_{21}x_1)}{m_2} - \frac{\partial U}{\partial x_1} \qquad \dot{x}_1 = \frac{p_1 - B_{12}x_2}{m_1}
$$

$$
\dot{p}_2 = \frac{B_{12}(p_1 - B_{12}x_2)}{m_1} - \frac{\partial U}{\partial x_2} \qquad \dot{x}_2 = \frac{p_2 - B_{21}x_1}{m_2}
$$

```
H = (p1 - B12 x2)^2/(2 m1) + (p2 - B21 x1)^2/(2 m2) + U[x1, x2];
-D[H, x1]
-D[H, x2]
D[H, p1]
D[H, p2]
```

$$\frac{B21\,(p2 - B21\,x1)}{m2} - U^{(1,0)}[x1, x2]$$

$$\frac{B12\,(p1 - B12\,x2)}{m1} - U^{(0,1)}[x1, x2]$$

$$\frac{p1 - B12\,x2}{m1}$$

$$\frac{p2 - B21\,x1}{m2}$$

(c)

$$\begin{aligned} B_{12}\dot{x}_1x_2 + B_{21}x_1\dot{x}_2 &= B_{12}\dot{x}_1x_2 + B_{21}x_1\dot{x}_2 + (B_{21}\dot{x}_1x_2 - B_{21}\dot{x}_1x_2) \\ &= B_{12}\dot{x}_1x_2 + (B_{21}x_1\dot{x}_2 + B_{21}\dot{x}_1x_2) - B_{21}\dot{x}_1x_2 \\ &= (B_{12} - B_{21})\,\dot{x}_1x_2 + \frac{d}{dt}\Lambda(x_1, x_2, t) \end{aligned}$$

The terms containing B_{12} and B_{21} can be written as a total time derivative plus a term depending only on $(B_{12} - B_{21})$, and so these terms effectively only depend on $(B_{12} - B_{21})$. The Hamiltonian, on the other hand, depends explicitly on both B_{12} and B_{21}.

Problem 3.

(a)

$$\begin{aligned} p_n &= \frac{\partial L}{\partial v_n} \\ &= \frac{mv_n}{\sqrt{1-\frac{v_n^2}{c^2}}} + \frac{e}{2c}\epsilon_{ijk}B_ix_j\delta_{kn} \\ &= \frac{mv_n}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{e}{2c}\epsilon_{ijn}B_ix_j \\ \implies \vec{p} &= \frac{m\vec{v}}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{e}{2c}\left(\vec{B}\times\vec{r}\right) \\ p_2 &= \frac{mv_2}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{e}{2c}\left(B_3x_1 - B_1x_3\right) \end{aligned}$$

$$\begin{aligned} \vec{p} &= \frac{m\vec{v}}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{e}{2c}\left(\vec{B}\times\vec{r}\right) \\ \implies \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2 &= \frac{m^2v_i^2}{1-\frac{v_i^2}{c^2}} \\ \implies m^2v_i^2 &= \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2 - \frac{v_i^2}{c^2}\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2 \\ \implies \frac{v_i^2}{c^2}\left(m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right) &= \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2 \\ \implies \frac{v_i^2}{c^2} &= \frac{\left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}{m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2} \\ \implies 1 - \frac{v_i^2}{c^2} &= \frac{m^2c^2}{m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2} \\ \implies \sqrt{1-\frac{v_i^2}{c^2}} &= \frac{mc}{\sqrt{m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}} \end{aligned}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v_i^2}{c^2}}} + \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)$$

$$\implies m\vec{v} = \sqrt{1-\frac{v_i^2}{c^2}}\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)$$

$$\implies \vec{v} = \frac{c\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}$$

$$v_3 = \frac{c\left(\vec{p}-\frac{e}{2c}\left(B_1x_2-B_2x_1\right)\right)}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}$$

$$D\left[-m\,c^2\sqrt{1-\frac{v^2}{c^2}},\ v\right]$$

$$\frac{m\,v}{\sqrt{1-\frac{v^2}{c^2}}}$$

(b)

$$
\begin{aligned}
H &= p_i v_i - L \\
&= p_i v_i + mc^2\sqrt{1-\frac{v_i^2}{c^2}} - \frac{e}{2c}\epsilon_{ijk}B_i x_j v_k \\
&= \vec{p}\cdot\frac{c\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}} + \frac{m^2c^3}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}} \\
&\quad - \frac{e}{2c}\epsilon_{ijk}B_i x_j \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(p_k - \frac{e}{2c}\epsilon_{klm}B_l x_m\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(\vec{p}^2 - \frac{e}{2c}\vec{p}\cdot\left(\vec{B}\times\vec{r}\right) + m^2c^2 - \frac{e}{2c}\vec{p}\cdot\left(\vec{B}\times\vec{r}\right) + \frac{e^2}{4c^2}\epsilon_{ijk}B_i x_j \epsilon_{klm}B_l x_m\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(m^2c^2 + \vec{p}^2 - 2\vec{p}\cdot\left(\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right) + \left(\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right) \\
&= \frac{c}{\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}}\left(m^2c^2 + \left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)
\end{aligned}
$$

$$
H = c\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2}
$$

(c)

$$
\begin{aligned}
H &= c\sqrt{m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2} \\
&= c\sqrt{m^2c^2+\vec{p}^2-\frac{e}{c}\epsilon_{ijk}p_i B_j x_k + \frac{e^2}{4c^2}\epsilon_{ijk}B_j x_k \epsilon_{ilm}B_l x_m}
\end{aligned}
$$

$$
\begin{aligned}
\dot{p}_n &= -\frac{\partial H}{\partial x_n} \\
&= -\frac{c}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(-\frac{e}{c}\epsilon_{ijn}p_i B_j + \frac{e^2}{4c^2}\epsilon_{ijk}B_j\epsilon_{ilm}B_l\left(\delta_{kn}x_m+\delta_{mn}x_k\right)\right) \\
&= -\frac{c}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\frac{e}{c}\epsilon_{ijn}B_i p_j + \frac{e^2}{4c^2}\left(\epsilon_{ijn}B_j\epsilon_{ilm}B_l x_m + \epsilon_{ijk}B_j x_k \epsilon_{iln}B_l\right)\right) \\
&= -\frac{c}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\frac{e}{c}\epsilon_{ijn}B_i p_j + \frac{e^2}{2c^2}\epsilon_{ijn}B_j\epsilon_{ilm}B_l x_m\right) \\
&= -\frac{c}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\frac{e}{c}\epsilon_{ijn}B_i p_j - \frac{e^2}{2c^2}\epsilon_{nji}B_j\epsilon_{ilm}B_l x_m\right) \\
&= -\frac{e}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\left(\vec{B}\times\vec{p}\right)_n - \frac{e}{2c}\left(\vec{B}\times\left(\vec{B}\times\vec{r}\right)\right)_n\right) \\
&= \frac{e}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\left(\vec{p}\times\vec{B}\right)_n + \frac{e}{2c}\left(\vec{B}\times\left(\vec{B}\times\vec{r}\right)\right)_n\right)
\end{aligned}
$$

$$
\implies \dot{\vec{p}} = \frac{e}{2}\left(m^2c^2+\left(\vec{p}-\frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(\left(\vec{p}\times\vec{B}\right) + \frac{e}{2c}\left(\vec{B}\times\left(\vec{B}\times\vec{r}\right)\right)\right)
$$

$$
\begin{aligned}
\dot{x}_n &= \frac{\partial H}{\partial p_n} \\
&= \frac{c}{2}\left(m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(2p_n - \frac{e}{c}\epsilon_{njk}B_jx_k\right) \\
&= \left(m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(cp_n + \frac{e}{2}\epsilon_{njk}x_jB_k\right) \\
\implies \dot{\vec{x}} &= \left(m^2c^2 + \left(\vec{p} - \frac{e}{2c}\left(\vec{B}\times\vec{r}\right)\right)^2\right)^{-\frac{1}{2}}\left(c\vec{p} + \frac{e}{2}\left(\vec{r}\times\vec{B}\right)\right)
\end{aligned}
$$