Pion-Pion Scattering in the Non-Linear Sigma Model

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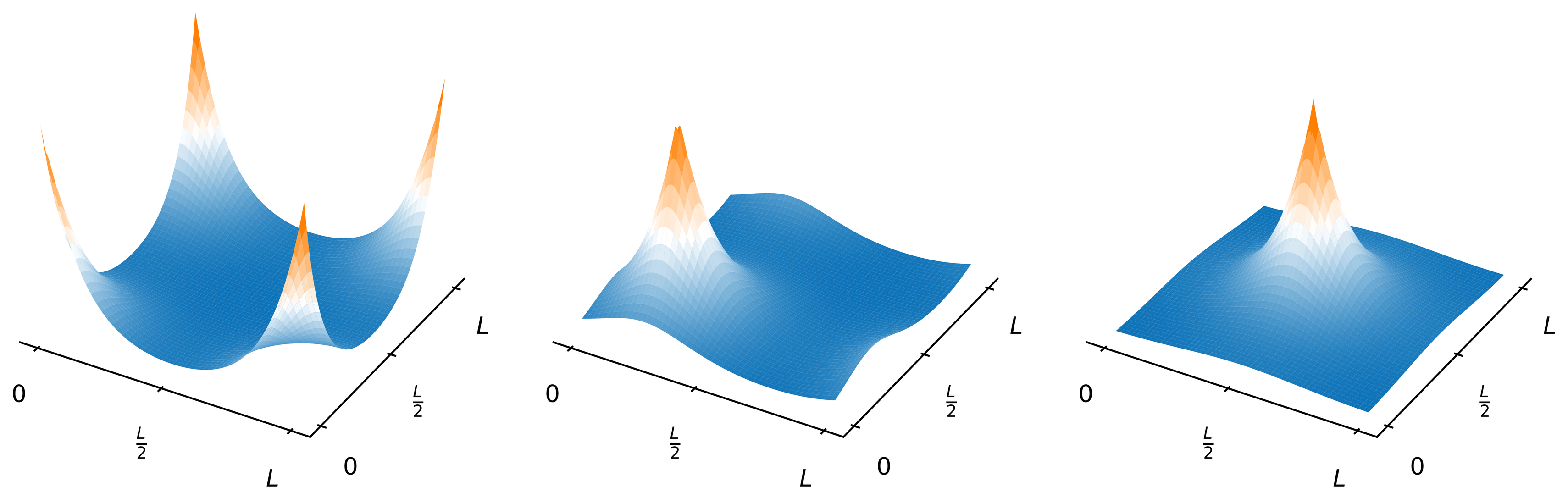
Background

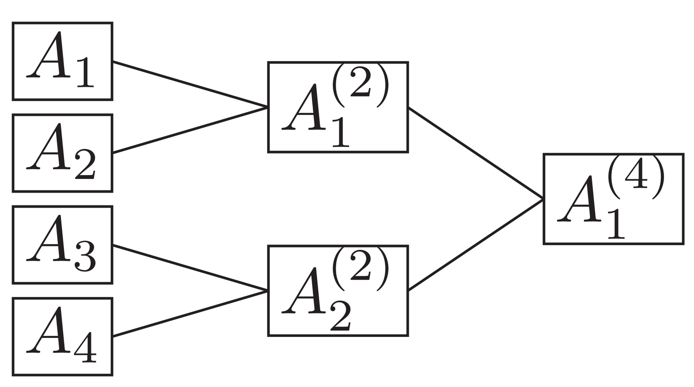
For a periodic lattice with grid spacing a and particle mass m, the two-point correlation function is given by [1]

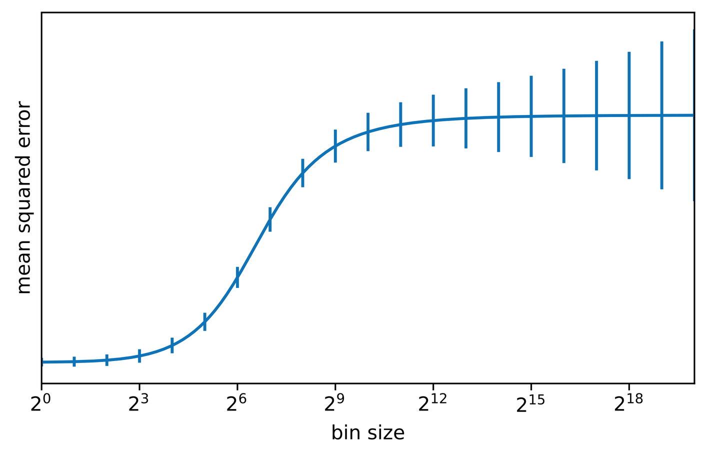
$$c(\delta) \sim e^{-am\,\delta} + e^{-am(T-\delta)}. \tag{1}$$

The dimensionless quantity am may be a parameter of the system (e.g. Klein-Gordon field) or can be determined by fitting calculated two-point correlations (e.g. non-linear sigma model).



The binning method [2] can be used to find the uncertainty in calculated observables of a system using Markov chain Monte Carlo algorithms.





The integrated autocorrelation time [2] for an estimator $\hat{\mu}$

$$\tau_{\hat{\mu}} = \frac{\mathrm{MSE}(\hat{\mu})}{\mathrm{MSE}_{\text{naïve}}(\hat{\mu})} = \frac{N}{N_{\mathrm{eff},\hat{\mu}}}$$

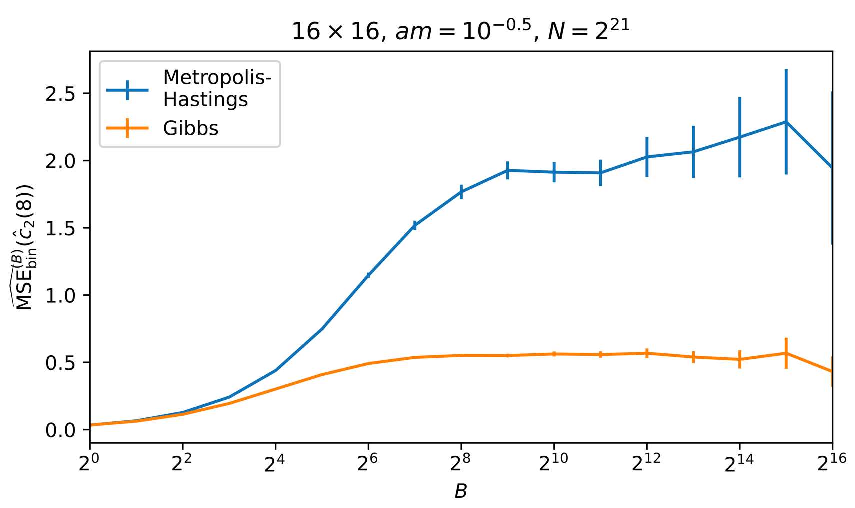
is given by the ratio of the true and “naïve” errors (i.e. assuming independent data), and gives rise to the notion of effective independence.

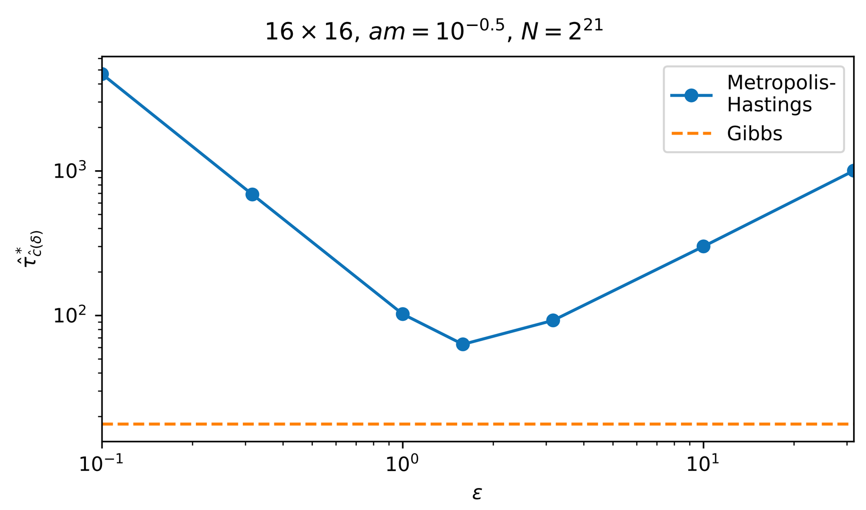
Klein-Gordon Theory

The action for Klein-Gordon theory on a lattice is given by [1]

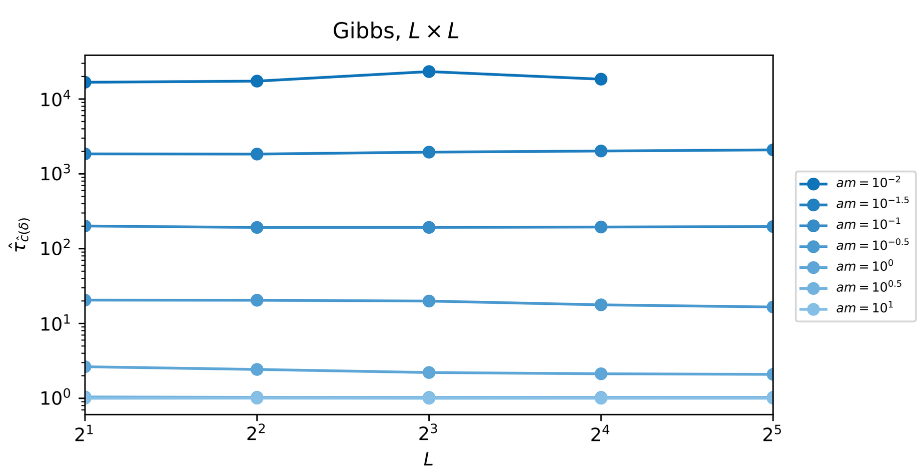
$$S(\phi) = \sum_{\mathbf{x}} \phi(\mathbf{x}) \left(\frac{2d + (am)^2}{2} \phi(\mathbf{x}) - \sum_{i=1}^{d} \phi(\mathbf{x} + \mathbf{e}_i) \right).$$

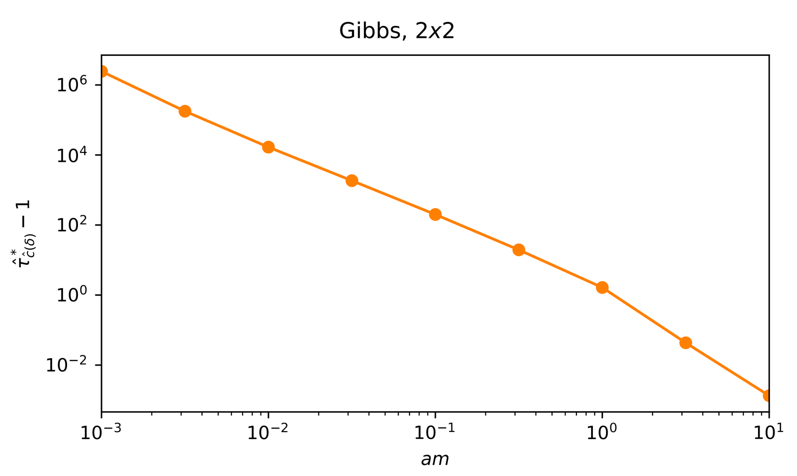
The Metropolis-Hastings algorithm and Gibbs sampler [3] were used to calculate two-point correlations for Klein-Gordon fields. The Gibbs sampler was found to be preferable due to its significantly weaker autocorrelation and the lack of an external parameter.





The integrated autocorrelation time for the two-point correlation function was found to be roughly independent of lattice size, and had an approximate power law relationship with am ($\tau_{\hat{c}(\delta)} \propto am^{-2}$ for $am < 1$).



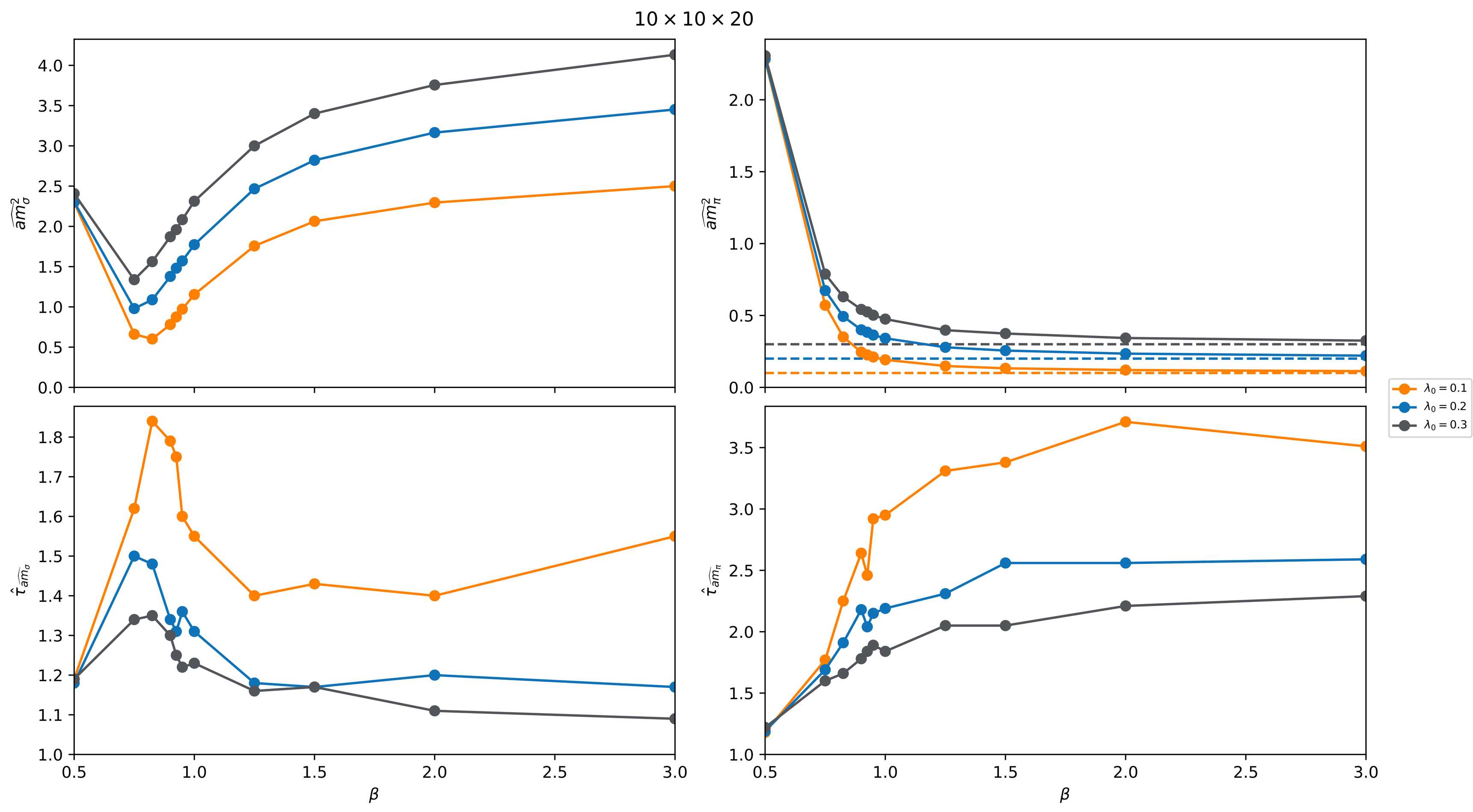


Non-Linear Sigma Model

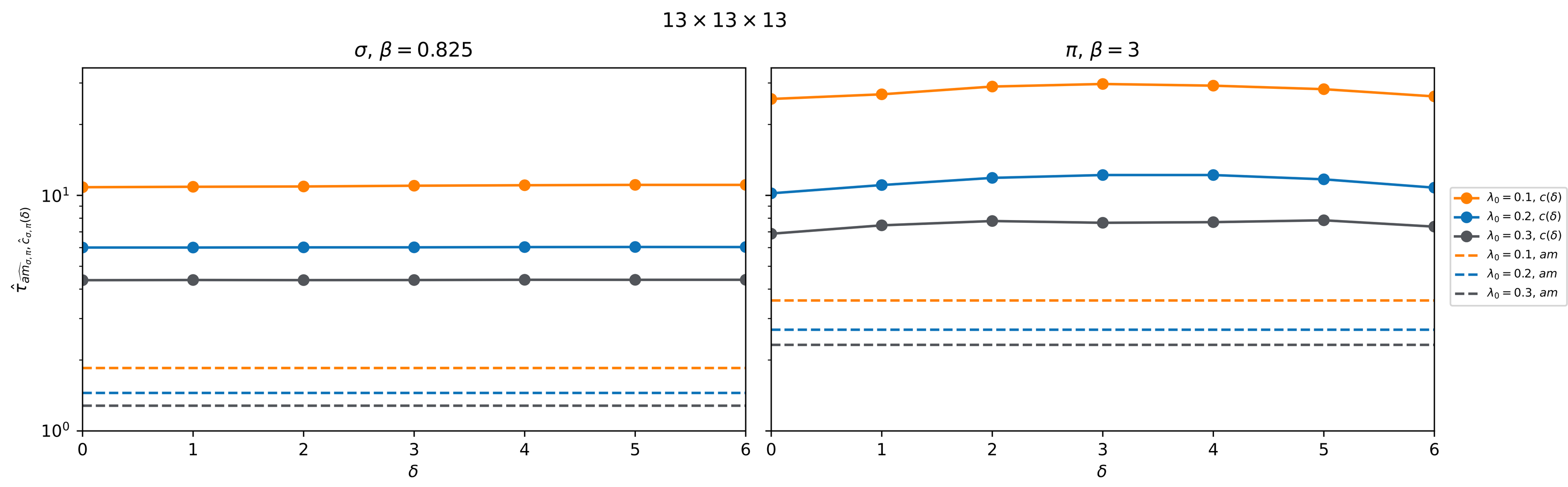
The action for the $SU(2)$ non-linear sigma model on a lattice is given by [4]

$$S(\boldsymbol{\phi}) = -\frac{\beta}{2} \sum_{\mathbf{x}} \mathrm{Tr}\left[\boldsymbol{\phi}(\mathbf{x}) \left(\lambda_0 \mathbb{I} + \sum_{i=1}^{d} \boldsymbol{\phi}^\dagger(\mathbf{x} + \mathbf{e}_i) \right) \right],$$
$$\boldsymbol{\phi} = (\sigma, \boldsymbol{\pi}) \in SU(2).$$

Two-point correlations were calculated for various β and λ_0. am_σ and am_π were estimated using the jackknife method [2] for eq. (1).



The behaviour of the σ and π masses were found to be notably different; am_σ has a λ_0-dependent minimum at which its autocorrelation peaks, whereas $(am_\pi)^2 \to \lambda_0$ and its autocorrelation is maximised for $\beta \to \infty$. The autocorrelation for the π masses was found to be generally stronger than that for the σ masses, i.e. $\tau_{\widehat{am}_\sigma} < \tau_{\widehat{am}_\pi}$. It was also discovered that the mass autocorrelation was appreciably weaker than that for the two-point correlation, i.e. $\tau_{\widehat{am}} < \tau_{\hat{c}(\delta)}$.



Similar results were obtained for various lattice sizes.

Conclusion

The study of Klein-Gordon theory in this project built a strong foundation for simulating the non-linear sigma model. This was done by

- debugging code via numerical and analytic comparison,
- improving efficiency by comparing algorithms and programmes, and
- finding relationships between autocorrelation and parameters.

The non-linear sigma model was then investigated, where two-point correlations were calculated and used to determine particle masses. The dependance of these masses on the system parameters was determined, and it was shown that the autocorrelation of an algorithm does not only depend on the system and its parameters, but also on the observable being calculated.

References

[1] H. J. Rothe, *Lattice Gauge Theories: An Introduction.* World Scientific Lecture Notes in Physics, Singapore: World Scientific Publishing Company, 3rd ed., 2005.

[2] B. A. Berg, *Markov Chain Monte Carlo Simulations and Their Statistical Analysis With Web-Based Fortran Code.* Singapore: World Scientific Publishing, 2004.

[3] S. M. Ross, *Simulation.* Cambridge: Academic Press, 5th ed., 2005.

[4] F. Gürsey, “On the Symmetries of Strong and Weak Interactions,” *Nuovo Cimento*, vol. 16, no. 2, pp. 230–240, 1960.