Senior Fresh Computational Physics Laboratory Report

Laboratory 3: Projectile Motion

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Contents

1 Introduction

The aim of this laboratory was to investigate the effects of air resistance on motion of a particle, and compare linear and quadratic resistances.

1.1 Motion in a Vacuum

For many cases, motion can be approximated to be in a vacuum, i.e. to have no air resistance. Examples of such cases are motion of a very heavy mass, or motion under very weak air resistance. In these cases, Newton's 2^{nd} Law

$$F = ma \tag{1}$$

can be rewritten for projectile motion. Since there is no acceleration in the x direction, and the only force acting on the particle is mg in the $-y$ direction, the equations

$$\frac{dv_x}{dt} = 0 \tag{2}$$

$$\frac{dv_y}{dt} = -g \tag{3}$$

can be derived for projectile motion in a vacuum. By integrating with respect to time t, the equations

$$v_x = v_{x0} \tag{4}$$

$$v_y = v_{y0} - gt \tag{5}$$

define the velocity of a particle in a vacuum, where x_0 and y_0 correspond to $t = 0$.

1.2 Motion Under Air Resistance

When the air resistance acting on a particle is non-negligible, the particle's weight is not the only force that can be considered to act on the particle. By using a Taylor Series to expand the resistive force, realising that the resistive force only exists when the particle is not at rest, and neglecting terms of order 3 and higher, the resistive force magnitude can be written[1]

$$f(v) = bv + cv^2, \tag{6}$$

where b and c are dependent on the medium through which the particle is travelling. Considering this force, along with the particle's weight, leads to a rewriting of equation 1 as

$$\frac{dv_x}{dt} = -\frac{b}{m}v_x - \frac{c}{m}\sqrt{v_x^2 + v_y^2}\,v_x \tag{7}$$

$$\frac{dv_y}{dt} = -g - \frac{b}{m}v_x - \frac{c}{m}\sqrt{v_x^2 + v_y^2}\,v_y \tag{8}$$

to describe the projectile motion of a particle in air. For small time increments Δt, the changes in velocity Δv can be written as

$$\Delta v_x = -\frac{b}{m}v_x\Delta t - \frac{c}{m}\sqrt{v_x^2 + v_y^2}\,v_x\Delta t \tag{9}$$

$$\Delta v_y = -g\Delta t - \frac{b}{m}v_x\Delta t - \frac{c}{m}\sqrt{v_x^2 + v_y^2}\,v_y\Delta t. \tag{10}$$

[1]Taylor, 2005

For negligible quadratic air resistance (i.e. $bv \gg cv^2$), the acceleration in the y direction has the numerical equation[2]

$$v_y(t) = \frac{mg}{b}\left(e^{-\frac{bt}{m}} - 1\right). \tag{11}$$

Taking the limit of this equation as $t \to \infty$ (or neglecting c and setting $\frac{dv_y}{dt} = 0$ in equation (6)) results in

$$V_T = -\frac{mg}{b}, \tag{12}$$

an expression for the terminal velocity of a falling particle experiencing linear air resistance.

For a spherical particle travelling through air, equation (4) can be written in terms of the particle's diameter D and velocity v as

$$f(Dv) = B\,Dv + C\,(Dv)^2, \tag{13}$$

with $b = B\,D$ and $c = C\,D$, and B and C known to be 1.6×10^{-4} N s m^{-1} and 0.25 N s^2 m^{-4}, respectively.[1]

2 Methodology

2.1 Exercise 1: Comparison of Linear and Quadratic Air Resistance

1. Functions were defined using equation (13) for total, linear and quadratic resistive force.

2. The ranges for which the linear and quadratic terms can be neglected were found using the following code:

```
neg = 0.9
quadratic_neglection = []
linear_neglection = []
DV_list = np.linspace(0., 0.06, 100001)
for i in DV_list:
    if i !=0:
        if f_linear(i) / f(i) > neg:
            quadratic_neglection.append(i)
        if f_quadratic(i) / f(i) > neg:
            linear_neglection.append(i)
```

   Here, `f`, `f_linear` and `f_quadratic` are the functions defined in Step 1., and the linear or quadratic term was deemed negligible if it was less than 10% of the total force.

3. The resistive force and the linear and quadratic contributions were plotted against diameter times velocity Dv for various ranges, showing the ranges for which the linear and quadratic terms are negligible.

4. Step 3. was repeated, showing the resistive force for various spherical objects travelling at certain speeds.

[2]TCD School of Physics, 2021

2.2 Exercise 2: Vertical Motion Under Air Resistance

1. The velocity of a spherical particle released from rest was plotted against time by creating a loop that repeatedly iterated equation (10), for a given diameter D, density ρ, and time increment Δt.

2. The error between the numerical equation (10) and analytical equation (11) was plotted against time.

3. Steps 2. and 3. were repeated for particles of various diameters.

4. The time for a spherical particle to fall 5 m was plotted against mass by repeatedly calculating this time for a particle of a given diameter and density, and calculating the corresponding mass.

2.3 Exercise 3: Projectile Motion Under Linear Air Resistance

1. Equations (9) and (10) were repeatedly iterated to calculate the velocity of a particle of a given diameter and density over time, for a given launch angle and speed. The quadratic term was neglected by setting $c = 0$.

2. The trajectory of this particle was plotted by multiplying each Δv_x and Δv_y by Δt to find the distance travelled during each time increment.

3. The angle at which the particle attained its maximum range was plotted as a function of mass. This was achieved by calculating the range of a particle of a given mass for various launch angles and finding the angle at which the maximum range was attained, and repeating for a range of masses.

2.4 Exercise 4: Projectile Motion Under Quadratic Air Resistance

1. Exercise 2 Steps 1. and 2. were repeated, setting $b = 0$ instead of c. This was carried out for various diameters, launch angles and launch speeds, and the plots for linear, quadratic and no air resistance were plotted on the same graph.

2. Exercise 2 Step 3. was repeated, considering the quadratic air resistance instead. Again, the resulting plots were plotted on the same graph for comparison.

3 Results

3.1 Exercise 1: Comparison of Linear and Quadratic Air Resistance

The quadratic term was found to be negligible up to 7.08×10^{-5} m^2 s^{-1}, and the linear term was found to be negligible for 5.76×10^{-3} m^2 s^{-1} and larger. In between these, neither term was small enough to be neglected. The following graphs of the resistive force were plotted for these ranges.

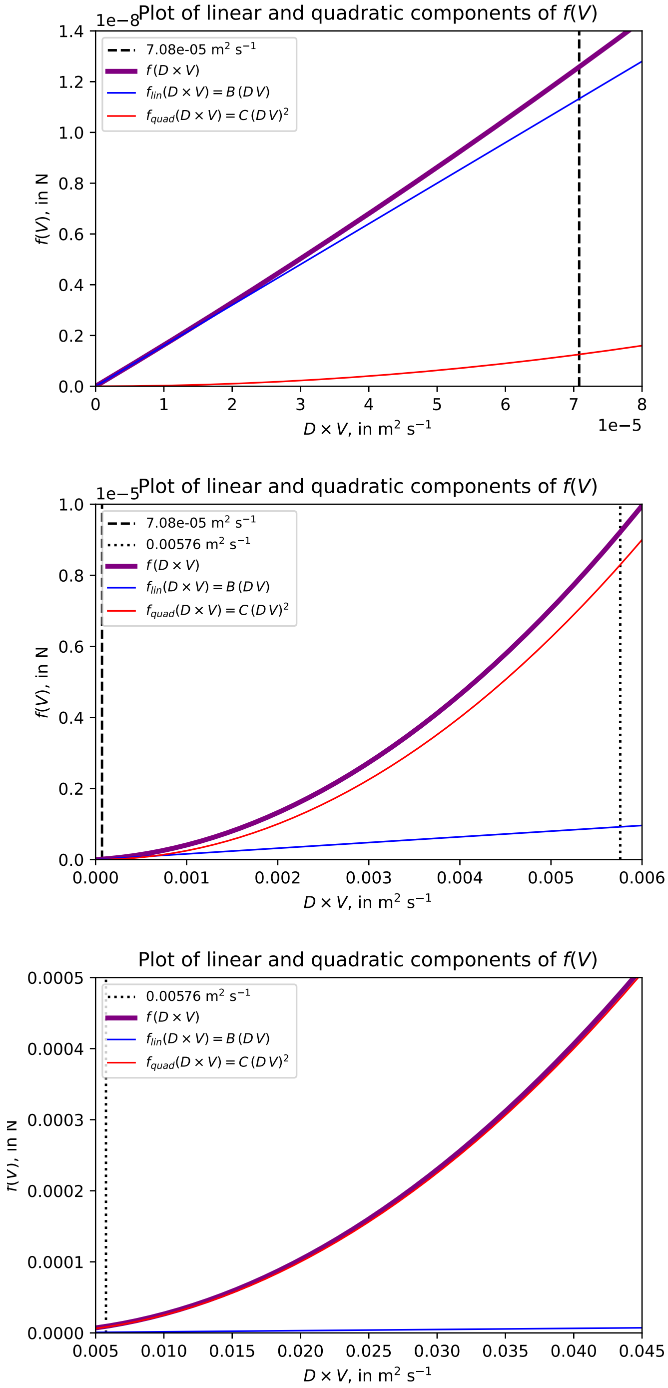

Figure 1: Graphs of the resistive force and its linear and quadratic contributions over ranges where the linear, quadratic or no term can be neglected.

The resistive force was also plotted for a baseball of diameter 7 cm travelling at 5 m s^{-1}, a drop of oil of diameter 1.5×10^{-6} m travelling at 5×10^{-5} m s^{-1}, and a raindrop of diameter 1 mm travelling at 1 m s^{-1}.

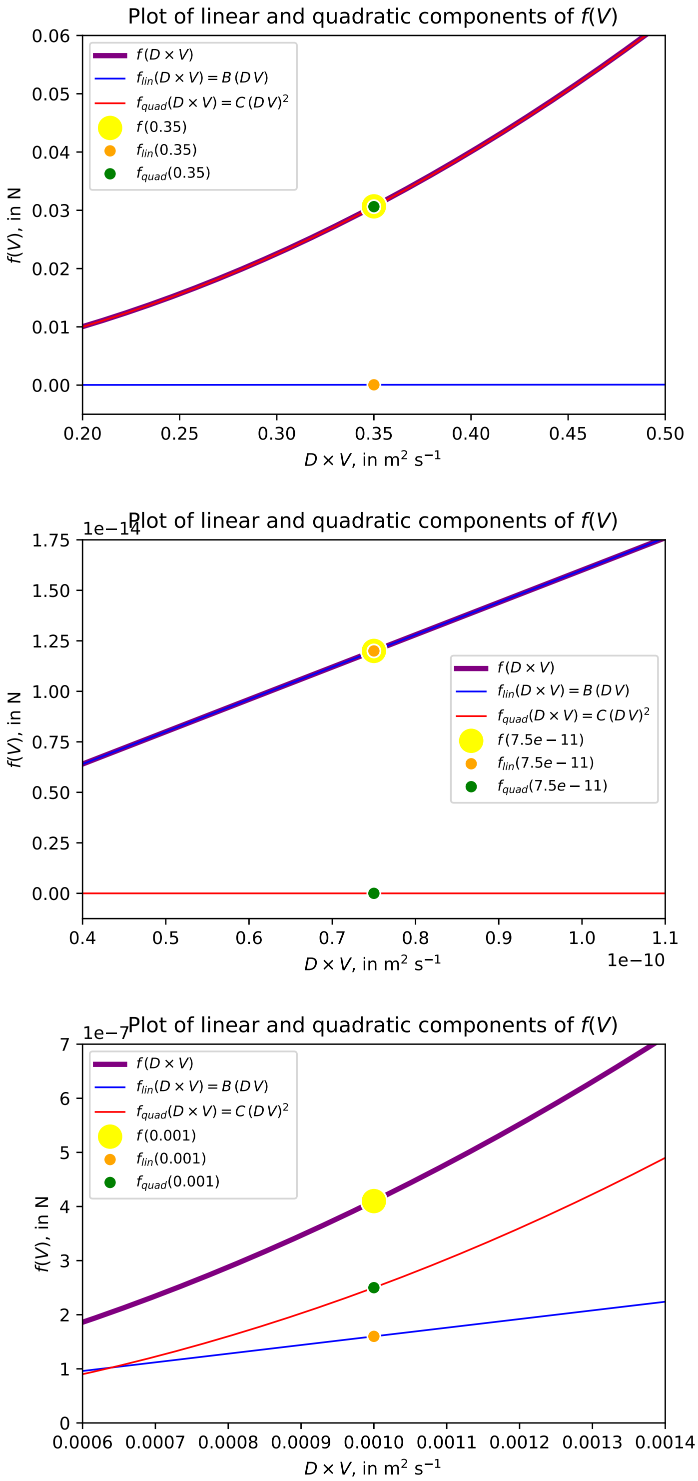

Figure 2: Graphs of the resistive force and its linear and quadratic contributions, showing the force acting on a baseball, oil drop, and raindrop, respectively.

From these graphs, and also using the ranges for which each term is negligible, it is clear that the linear term can be neglected for the baseball, the quadratic term can be neglected for the oil drop, and neither term can be neglected for the raindrop.

3.2 Exercise 2: Vertical Motion Under Air Resistance

For this exercise, the particle density ρ was kept as a constant 2×10^3 kg m^{-3}. The following data was obtained.

D	10^{-5} m	10^{-4} m
b	1.6×10^{-9} kg s^{-1}	1.6×10^{-8} kg s^{-1}
m	1.05×10^{-12} kg	1.05×10^{-9} kg
$\frac{b}{m}$	1528 s^{-1}	15.28 s^{-1}
c	2.5×10^{-11} kg m^{-1}	2.5×10^{-9} kg m^{-1}
V_T	0.00642 m s^{-1}	0.642 m s^{-1}
bV_T	1.03×10^{-11} N	1.03×10^{-8}
cV_T^2	1.03×10^{-15} N	1.03×10^{-9} N
$\frac{bV_T}{cV_T^2}$	1000	10

Figure 3: Data obtained for given density 2×10^3 kg m^{-3} and two diameters.

From this, it can be seen that an increase in mass alone is equivalent to a decrease in the linear coefficient b, and thus a decrease in the linear air resistance.

It can also be seen that, for small enough diameters (and thus small enough masses), the quadratic air resistance can be neglected, as was shown in Exercise 1.

The following graphs of velocity against time, and the corresponding error between the analytical and numerical values, were plotted for these diameters.

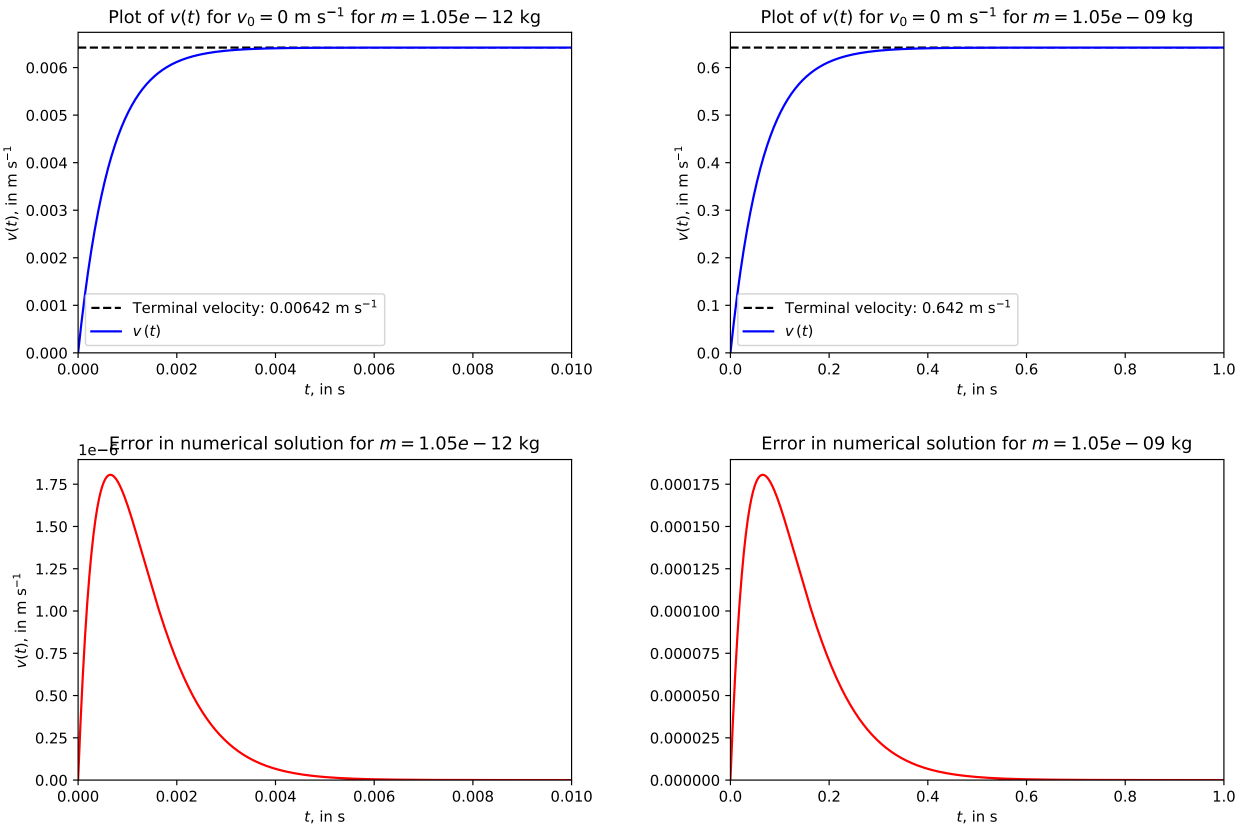

Figure 4: Plot of velocity of falling particle over time, alongside the error of the numerical results, for masses 1.05×10^{-12} kg and 1.05×10^{-9} kg, corresponding to density 2×10^3 kg m^{-3}, and diameters 10^{-5} m and 10^{-4} m.

These graphs are almost identical, with the main difference being the scaling of the axes. There is a sharp spike in the difference between numerical and analytical values near the start of each plot, which quickly asymptotes to 0 again. This is due to the fact that there is a relatively large error early on, but once the particle is travelling at its terminal velocity, there would be very negligible error between the numerical and analytical velocity.

The following graph of time to fall 5 m against mass was obtained.

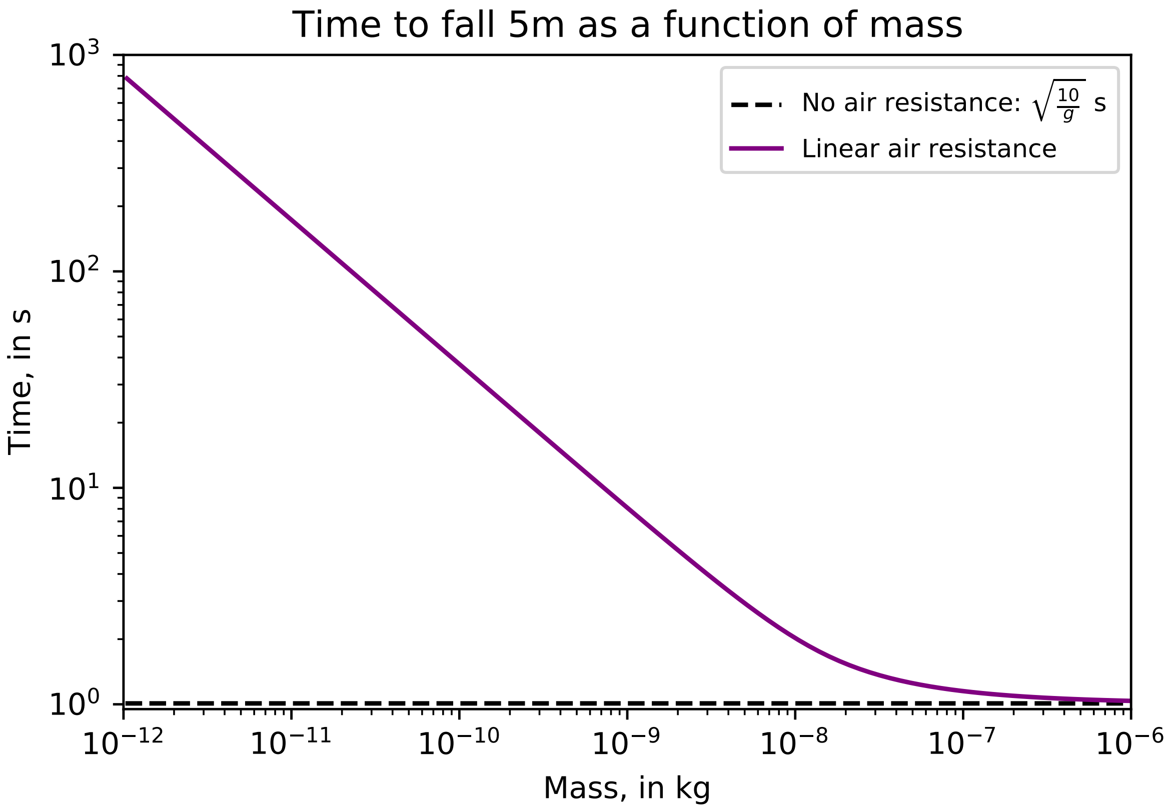

Figure 5: Plot of time for a spherical particle of density 2×10^3 kg m^{-3} to fall 5 m, as a function of mass. Note the logarithmic scale on both axes.

It can be seen from this graph that very small masses take a long time to fall to the ground. This is due to the fact that small masses reach their small terminal velocity very quickly (as shown in Figure 4), and so travel at this low speed for most of their fall.

This graph also shows that, as mass increases, the time to fall 5 m asymptotes to approximately 1 s. This is because, for a large enough mass, the linear air resistance is negligible, and so the time to fall 5 m is very similar to that in a vacuum. From the distance equation $s = ut + \frac{1}{2}at^2$, it can be found that the time for a particle to fall 5 m in a vacuum is $\sqrt{\frac{10}{g}}$ s, or approximately 1 s.

3.3 Exercise 3: Projectile Motion Under Linear Air Resistance

The following trajectory for $\rho = 2 \times 10^3$ kg m^{-3} and $D = 10^{-4}$ m, launch angle $\theta = \frac{\pi}{4}$ and initial speed $u = 1$ m s^{-1} under linear and no air resistance was obtained.

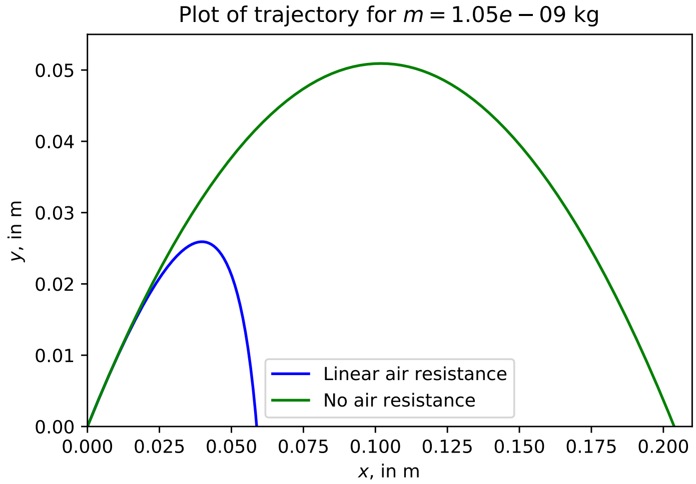

Figure 6: Plot of trajectory of spherical particle of density 2×10^3 kg m^{-3} and diameter 10^{-4} m.

The large difference in the linear air resistance trajectory and vacuum trajectory arises due to the small size of the mass, and shows that air resistance cannot always be neglected.

The following graph of optimum launch angle against mass was plotted, for $u = 1$ m s^{-1}.

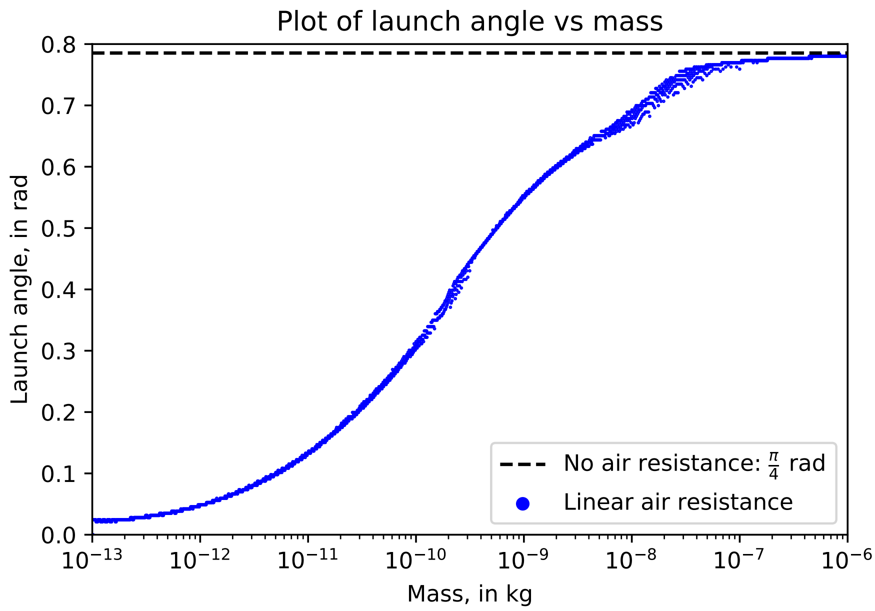

Figure 7: Plot of optimum launch angle against mass. Note the logarithmic x axis.

For very small masses, the optimum angle is very close to horizontal. This is because the ratio $\frac{b}{m}$ is much higher, and so there is a relatively strong force acting on the mass. The optimum angle would thus be very low, so the mass is not slowed down as much by the resistive force.

The graph also shows that, as mass increases, the optimal launch angle asymptotes to approximately $\frac{\pi}{4}$ rad. This is because, for a large enough mass, the linear air resistance is negligible, and so the optimal launch angle is very similar to that in a vacuum, which is $45° = \frac{\pi}{4}$ rad.

3.4 Exercise 4: Projectile Motion Under Quadratic Air Resistance

The following trajectories were obtained for $u = 1$ m s $^{-1}$, diameters 10^{-4} m, 2×10^{-4}m, 10^{-3} m, and 10^{-1} m, and launch angles $\frac{\pi}{8}$ rad, $\frac{\pi}{4}$ rad, and $\frac{3\pi}{8}$ rad.

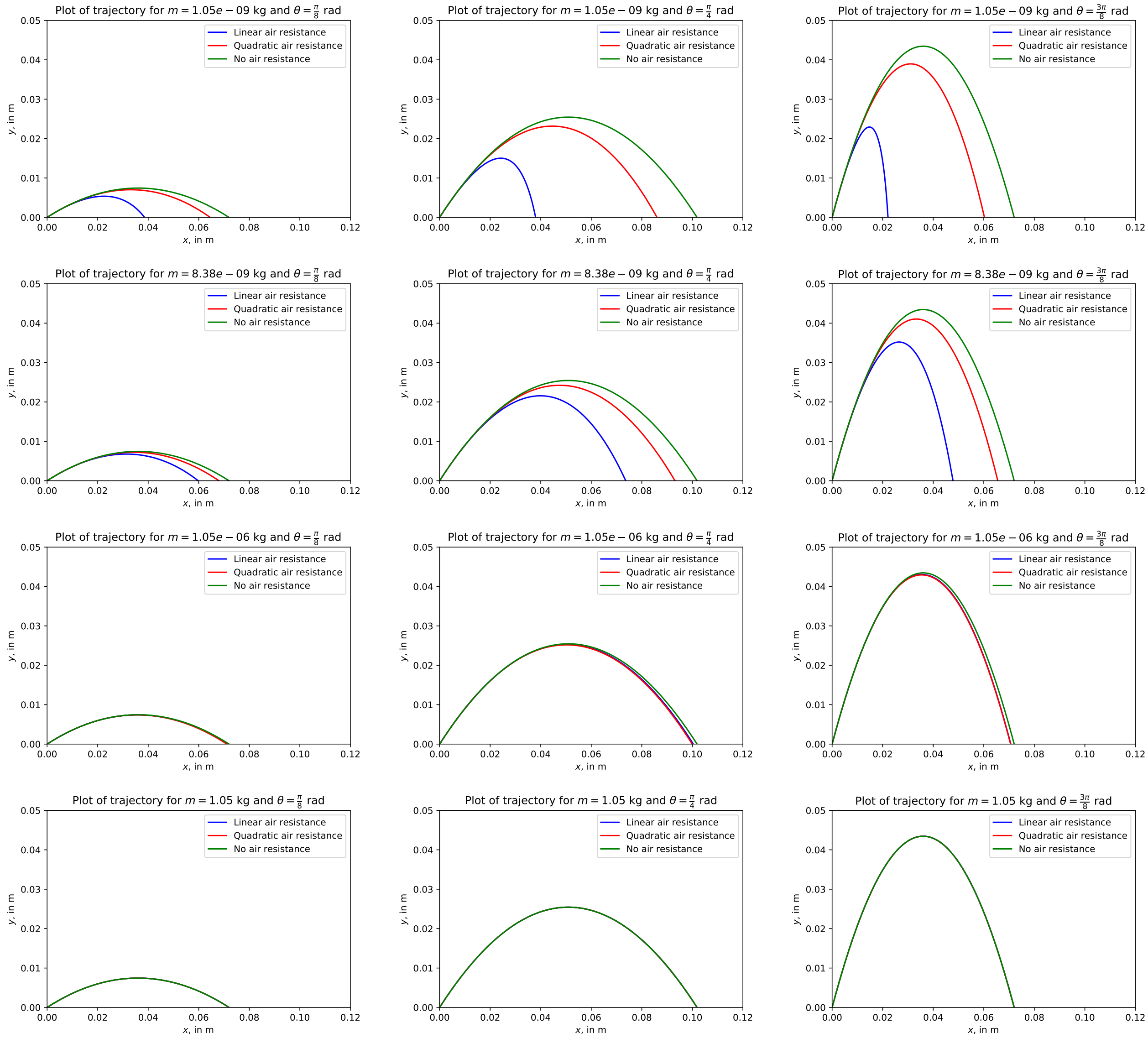

Figure 8: Trajectories of particles of various launch angles and diameters.

It can easily be seen from these graphs that, as the mass of the particle is increased (but the initial velocity is kept constant), the motion of the particle is similar to that in a vacuum, as the trajectory gets closer to that in a vacuum.

For masses 1.05×10^{-9} kg and 8.38×10^{-9} kg, the trajectory is noticably shorter, as the linear air resistance is stronger than the quadratic air resistance. This lines up with the discovery in Exercises 1 and 2 that the quadratic term is negligible for very small masses.

It can also be seen from the trajectories for the 1.05×10^{-9} kg mass that, for linear air resistance, a launch angle of $\frac{\pi}{4}$ rad does not correspond to a maximum range, as the range of the particle is slightly larger for $\theta = \frac{\pi}{8}$ than it is for $\theta = \frac{\pi}{4}$. This supports the graph of optimal launch angle against mass in Figure 7.

The following graphs of optimum launch angle against mass were plotted, for $u = 1$ m s^{-1}.

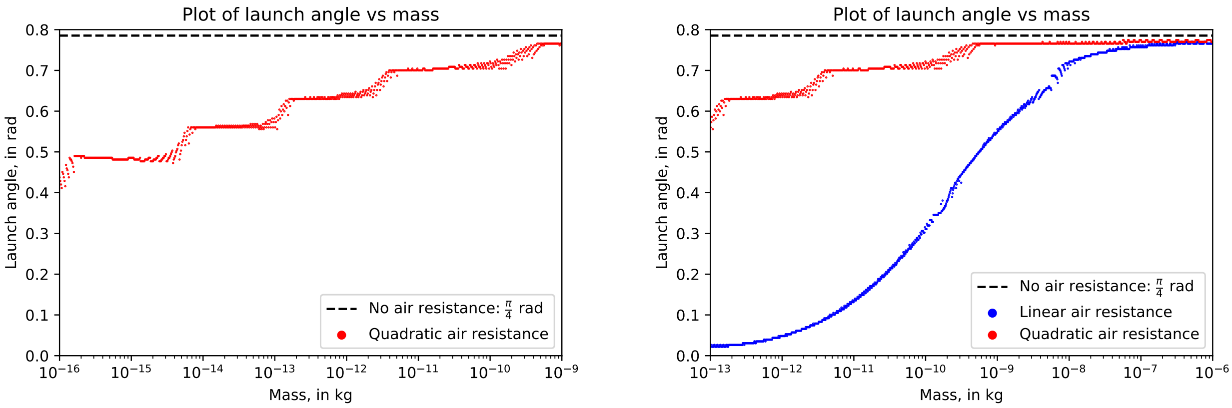

Figure 9: Plot of optimum launch angle against mass, for linear and quadratic air resistance. Note the logarithmic x axes.

As the mass decreases, the optimum launch angle gradually decreases. As before, this is due to the increase in relative force acting on the mass. This decrease is not as sharp as that for linear air resistance, however. This is due to the fact that, for smaller masses, the linear air resistance is much larger than the quadratic air resistance.

As with the linear air resistance, the optimal launch angle for quadratic air resistance asymptotes to $\frac{\pi}{4}$ rad as mass increases, due to the similarity to motion in a vacuum.

4 Conclusions

For most everyday cases, motion under air resistance is similar enough to motion in a vacuum, and so air resistance can be neglected. When particles of smaller size or mass are considered, however, air resistance can greatly affect motion.

If the air resistance force is considered to only have a linear and quadratic term, there are different ranges for which each term can and cannot be neglected. The quadratic air resistance can only be neglected for $Dv < 7.08 \times 10^{-5}$ m^2 s^{-1}, and the linear air resistance can only be neglected for $Dv > 5.76 \times 10^{-3}$ m^2 s^{-1}, where neglection is considered reasonable for values less than 10% of the total force.

It was shown that a decrease in mass results in a much stronger relative force, and an increase in mass leads to motion similar to that in a vacuum. By varying the mass, this was shown in numerous ways, including investigating the force coefficients directly, calculating time to fall a given height, and finding the launch angle at which a particle attains its maximum range.

5 References & Appendix

1: J. R. Taylor, *Classical Mechanics*, University Science Books, Huntington, 2005.

2: TCD School of Physics, provided laboratory instructions, 2021.

All code and figures used in this laboratory can be found here:
`https://github.com/campioru/SF_Lab_3`