Laboratory 2: The Nonlinear Pendulum

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Contents

1 Introduction

The aim of this laboratory was to investigate the motion of a pendulum using two different methods: the trapezoidal rule and the Runge-Kutta method.

1.1 Equation of Motion of a Pendulum

For a pendulum of length L, the arc length s travelled by the bob can be written in terms of the angle subtended by the pendulum θ as

$$s = L\,\theta,$$

with θ measured in radians. Since L remains constant, the acceleration of the bob can be written as

$$a \equiv \frac{d^2 s}{dt^2} = L\,\frac{d^2\theta}{dt^2}.$$

The force acting on the bob of mass m is its weight

$$F_{\text{weight}} = m\,g.$$

Since the component of this force parallel to the pendulum is balanced by tension in the pendulum, only the perpendicular component

$$F_{\text{weight},\perp} = mg\sin\theta$$

needs to be considered. If the pendulum experiences damping, a force proportional to $\omega \equiv \dfrac{d\theta}{dt}$ must be considered, and is written

$$F_{\text{damping}} = -b\,\omega,$$

where b is the damping coefficient. A periodic driven force acting on the pendulum can also be considered, and is written

$$F_{\text{driven}} = F_0\cos(\phi t),$$

where F_0 is the amplitude of the driving force and ϕ is the frequency of the force. Using Newton's 2^{nd} Law, assuming the mass is constant, leads to

$$\begin{aligned} ma &= \sum F \\ mL\,\frac{d^2\theta}{dt^2} &= -mg\sin\theta - b\,\omega + F_0\cos(\phi t) \\ f(\theta,\omega,t) \equiv \frac{d^2\theta}{dt^2} &= -\frac{g}{L}\sin\theta - k\,\omega + A\cos(\phi t), \end{aligned} \tag{1}$$

where $k \equiv \dfrac{b}{mL}$ and $A \equiv \dfrac{F_0}{mL}$. This is the most general form of the equation of motion of a pendulum. One could approximate the motion linearly for small angles by replacing $\sin\theta$ with θ, or investigate undamped or undriven motion by setting $k = 0$ or $A = 0$ respectively.

1.2 Trapezoidal Rule

Although there are not many methods to analytically solve ordinary differential equations, there are various numerical methods available. One such method is the trapezoidal rule.[1] This method relies on the geometry of the area under a curve, as well as the Taylor series expansion. The following expressions

$$\theta_{n+1} = \theta_n + \omega_n \frac{\Delta t}{2} + (\omega_n + f(\theta_n, \omega_n, t)\Delta t)\,\frac{\Delta t}{2} \tag{2}$$

$$\omega_{n+1} = \omega_n + f(\theta_n, \omega_n, t)\,\frac{\Delta t}{2} + f(\theta_{n+1}, \omega_n + f(\theta_n, \omega_n, t)\Delta t, t + \Delta t)\,\frac{\Delta t}{2}. \tag{3}$$

can be derived using this method, where $\theta_{n+1} \equiv \theta(t_{n+1})$ and $\omega_{n+1} \equiv \omega(t_{n+1})$ are calculated iteratively, Δt is the time step, and f is defined as in equation (1). As the expressions for θ_{n+1} and ω_{n+1} imply, this method is accurate up to 2^{nd} order in time. To use this method, initial conditions θ_0, ω_0, and t_0 must be defined. This method was used in Exercises 1, 2 and 3 to approximate the motion of an undamped, undriven pendulum.

1.3 Runge-Kutta Method

The Runge-Kutta methods[1] are a group of methods used to numerically solve ordinary differential equations. The trapezoidal rule is an example of one of these methods. Another example is the fourth-order Runge-Kutta method (also simply called the Runge-Kutta method) which, as the name implies, is accurate up to fourth order in time. The following expressions are used for θ_{n+1} and ω_{n+1}:

$$k_{1a} = \omega_n \, \Delta t \qquad k_{3a} = \left(\omega_n + \frac{k_{2b}}{2}\right)\Delta t$$

$$k_{1b} = f(\theta_n, \omega_n, t)\, \Delta t \qquad k_{3b} = f\left(\theta_n + \frac{k_{2a}}{2}, \omega_n + \frac{k_{2b}}{2}, t + \frac{\Delta t}{2}\right)\Delta t$$

$$k_{2a} = \left(\omega_n + \frac{k_{1b}}{2}\right)\Delta t \qquad k_{4a} = (\omega_n + k_{3b})\,\Delta t$$

$$k_{2b} = f\left(\theta_n + \frac{k_{1a}}{2}, \omega_n + \frac{k_{1b}}{2}, t + \frac{\Delta t}{2}\right)\Delta t \qquad k_{4b} = f(\theta_n + k_{3a}, \omega_n + k_{3b}, t + \Delta t)\,\Delta t$$

$$\theta_{n+1} = \theta_n + \frac{1}{6}\left(k_{1a} + 2k_{2a} + 2k_{3a} + k_{4a}\right) \tag{4}$$

$$\omega_{n+1} = \omega_n + \frac{1}{6}\left(k_{1b} + 2k_{2b} + 2k_{3b} + k_{4b}\right) \tag{5}$$

The Runge-Kutta method was used in Exercises 3, 4 and 5 to approximate the motion of various types of pendulums.

[1] Atkinson, 1989

1.4 Phase Space

Once $\theta(t)$ and $\omega(t)$ are found, they can be plotted against each other to illustrate the so-called phase space. One method of identifying if motion is periodic or not is by investigating the phase portrait of the motion. If the motion is periodic, the phase portrait will be a closed loop. For non-periodic motion, this loop will never become closed. If the motion is driven but periodic, the phase portrait will only be closed once the motion is no longer transient, but in its steady-state phase. In Exercise 5, the phase portraits of the pendulum were plotted for different initial conditions.

2 Methodology

2.1 Exercise 1: Linear Pendulum

1. A function was defined using equation (1) for linear, undamped, undriven motion by setting k and A to 0, and replacing $\sin\theta$ with θ.

2. θ_0, ω_0, Δt, the number of desired iteration steps, and the starting time t_0 were initialised.

3. To perform the trapezoidal rule, a `for` loop was made to iterate equations (2) and (3) over the desired number of iteration steps, appending each θ_{n+1} and ω_{n+1} to separate lists. A list was also made for the corresponding time increments. This was achieved using the following code:

```
theta_array = [theta]
omega_array = [omega]
t_array = [t]
for i in range(timesteps):
    k1a = omega * dt
    k1b = f(theta, omega, t) * dt
    k2a = (omega + k1b) * dt
    k2b = f(theta + k1a, omega + k1b, t + dt) * dt
    theta += (k1a + k2a)/2
    omega += (k1b + k2b )/2
    t += dt
    theta_array.append(theta)
    omega_array.append(omega)
    t_array.append(t)
```

   Here, `f` is the function defined in Step 1., `theta`, `omega` and `t` are θ_0, ω_0 and t_0, respectively, and `timesteps` and `dt` are the desired number of iteration steps and Δt, respectively.

4. The lists obtained in Step 3. were plotted against each other to plot graphs of $\theta(t)$ and $\omega(t)$.

5. Steps 2. - 4. were repeated for various values of θ_0 and ω_0.

2.2 Exercise 2: Nonlinear Pendulum - Trapezoidal Rule

1. A function was defined as in Exercise 1 Step 1., without the replacement of $\sin\theta$.
2. Exercise 1 Step 5. was repeated for this function. The graphs obtained using each function were plotted on the same graphs to compare differences.

2.3 Exercise 3: Nonlinear Pendulum - Runge-Kutta Method

1. The function in Exercise 2 Step 1. was again defined, and θ_0, ω_0, Δt, the number of desired iteration steps, and t_0 were initialised.
2. To perform the Runge-Kutta method, a `for` loop was made to iterate equations (4) and (5) over the desired number of iteration steps, appending each θ_{n+1} and ω_{n+1} to separate lists. A list was also made for the corresponding time increments. This was achieved using the following code:

```
theta_array = [theta]
omega_array = [omega]
t_array = [t]
for i in range(timesteps):
    k1a = omega * dt
    k1b = f(theta, omega, t) * dt
    k2a = (omega + k1b/2) * dt
    k2b = f(theta + k1a/2, omega + k1b/2, t + dt/2) * dt
    k3a = (omega + k2b/2) * dt
    k3b = f(theta + k2a/2, omega + k2b/2, t + dt/2) * dt
    k4a = (omega + k3b) * dt
    k4b = f(theta + k3a, omega + k3b, t + dt) * dt
    theta += (k1a + 2 * k2a + 2 * k3a + k4a)/6
    omega += (k1b + 2 * k2b + 2 * k3b + k4b)/6
    t += dt
    theta_array.append(theta)
    omega_array.append(omega)
    t_array.append(t)
```

3. The graphs of $\theta(t)$ obtained using the trapezoidal rule and using the Runge-Kutta method were plotted on the same graph to compare differences.

2.4 Exercise 4: Damped Pendulum

1. A function was defined using equation (1) for nonlinear, damped, undriven motion by setting A to 0. θ_0, ω_0, Δt, the number of desired iteration steps, t_0, and k were initialised.
2. The `for` loop used in Exercise 3 Step 2. to perform the Runge-Kutta method was again used for this function, and $\theta(t)$ and $\omega(t)$ were plotted.

2.5 Exercise 5: Damped, Driven Pendulum

1. A function was defined using equation (1) for nonlinear, damped, driven motion. θ_0, ω_0, Δt, the number of desired iteration steps, t_0, and k were initialised.

2. The `for` loop used in Exercise 3 Step 2. was again used for this function to find $\theta(t)$ and $\omega(t)$. Since θ and $\theta + 2\pi$ correspond to the same position of the pendulum, the following code was used to change any value of θ outside of the range $[-\pi, \pi]$:

```
for theta in theta_array:
    if theta < -np.pi + a or theta > np.pi + a:
        theta -= 2 * np.pi * np.abs(theta)/theta
```

   Here, `a` is a parameter used to shift the allowed range to $[-\pi + a, \pi + a]$.

3. Phase portraits for various values of A were created by plotting ω against θ.

3 Results

For convenience, the ratio $\frac{g}{L}$ was set to 1 for each exercise.

3.1 Exercise 1: Linear Pendulum

The following graphs of $\theta(t)$ and $\omega(t)$ of the linear pendulum were plotted for various values of θ_0 and ω_0:

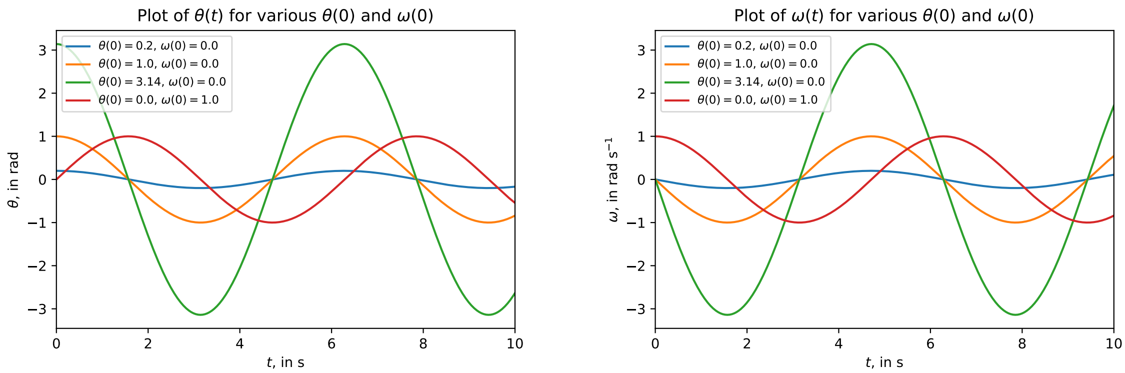

Figure 1: Graphs of $\theta(t)$ and $\omega(t)$ of the linear pendulum for various values of θ_0 and ω_0.

3.2 Exercise 2: Nonlinear Pendulum - Trapezoidal Rule

The following graphs of $\theta(t)$ and $\omega(t)$ of the linear and nonlinear pendulum were plotted for various values of θ_0 and ω_0:

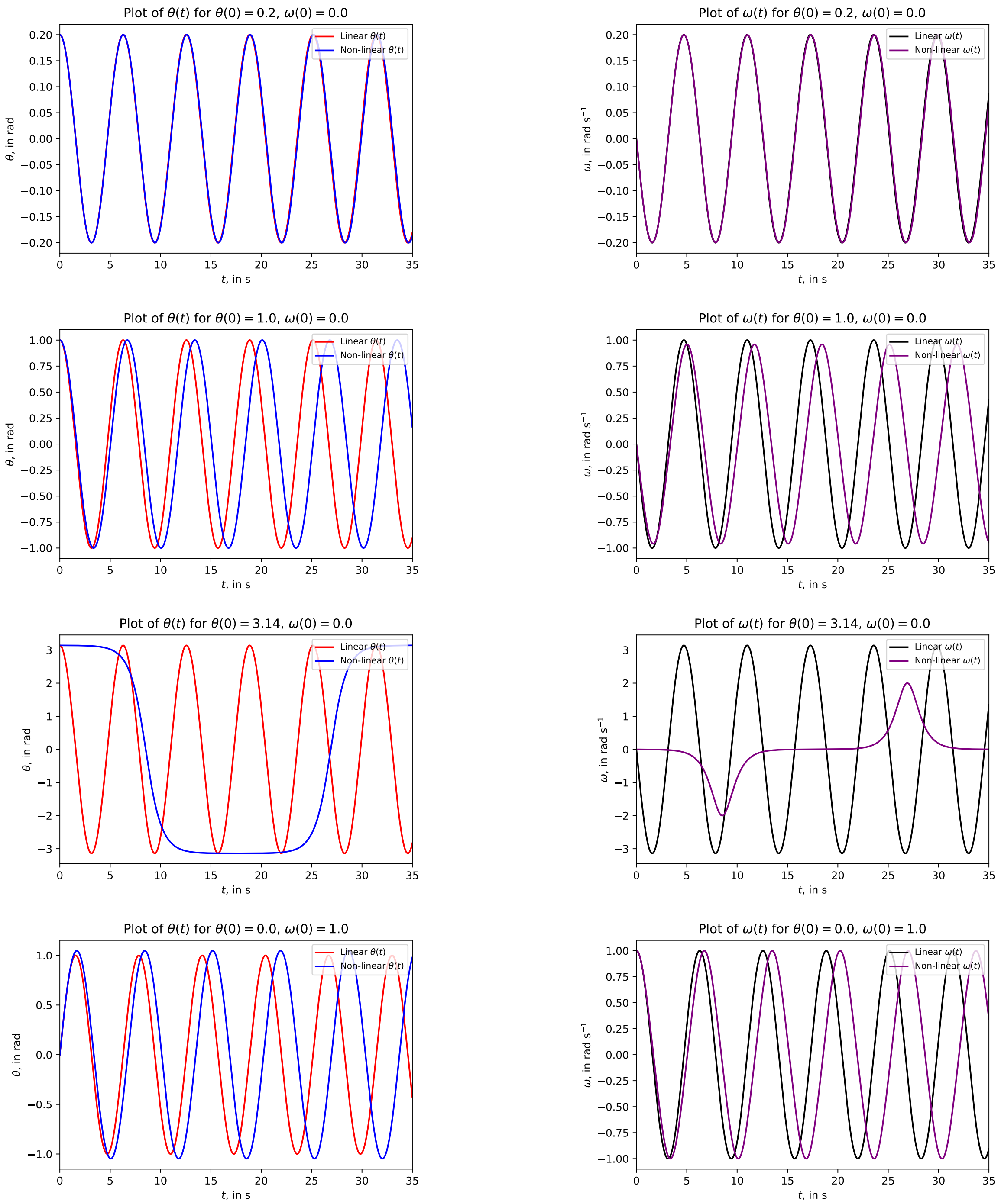

Figure 2: Graphs of $\theta(t)$ and $\omega(t)$ of the linear and nonlinear pendulum for various values of θ_0 and ω_0.

3.3 Exercise 3: Nonlinear Pendulum - Runge-Kutta Method

The following graph of $\theta(t)$ of the nonlinear pendulum was plotted using both the trapezoidal rule and Runge-Kutta method, for initial conditions $\theta_0 = 3.14$ and $\omega_0 = 0.0$:

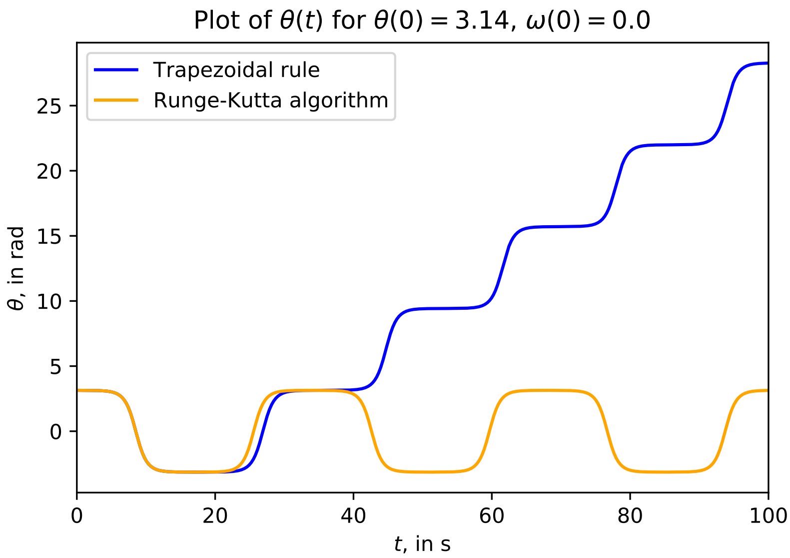

Figure 3: Graph of $\theta(t)$ of the nonlinear pendulum calculated using the trapezoidal rule and Runge-Kutta method, for $\theta_0 = 3.14$ and $\omega_0 = 0.0$.

3.4 Exercise 4: Damped Pendulum

The following graphs of $\theta(t)$ and $\omega(t)$ of the damped pendulum were plotted for $\theta_0 = 3.0$ and $\omega_0 = 0.0$, with $k = 0.5$:

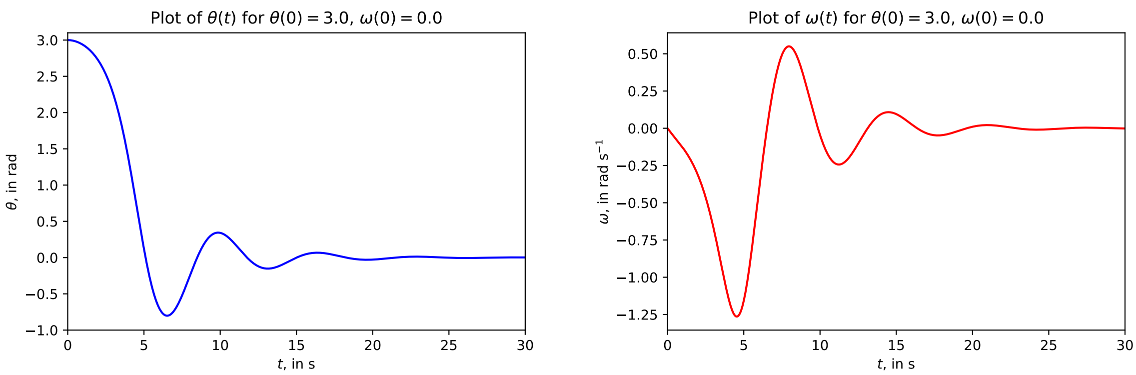

Figure 4: Graphs of $\theta(t)$ and $\omega(t)$ of the damped pendulum for $\theta_0 = 3.0$ and $\omega_0 = 0.0$, with $k = 0.5$.

3.5 Exercise 5: Damped, Driven Pendulum

The following phase portraits were plotted for the damped, driven pendulum for various values of A, with $\theta_0 = 1.0$, $\omega_0 = 0.0$, and time range 100-250 seconds:

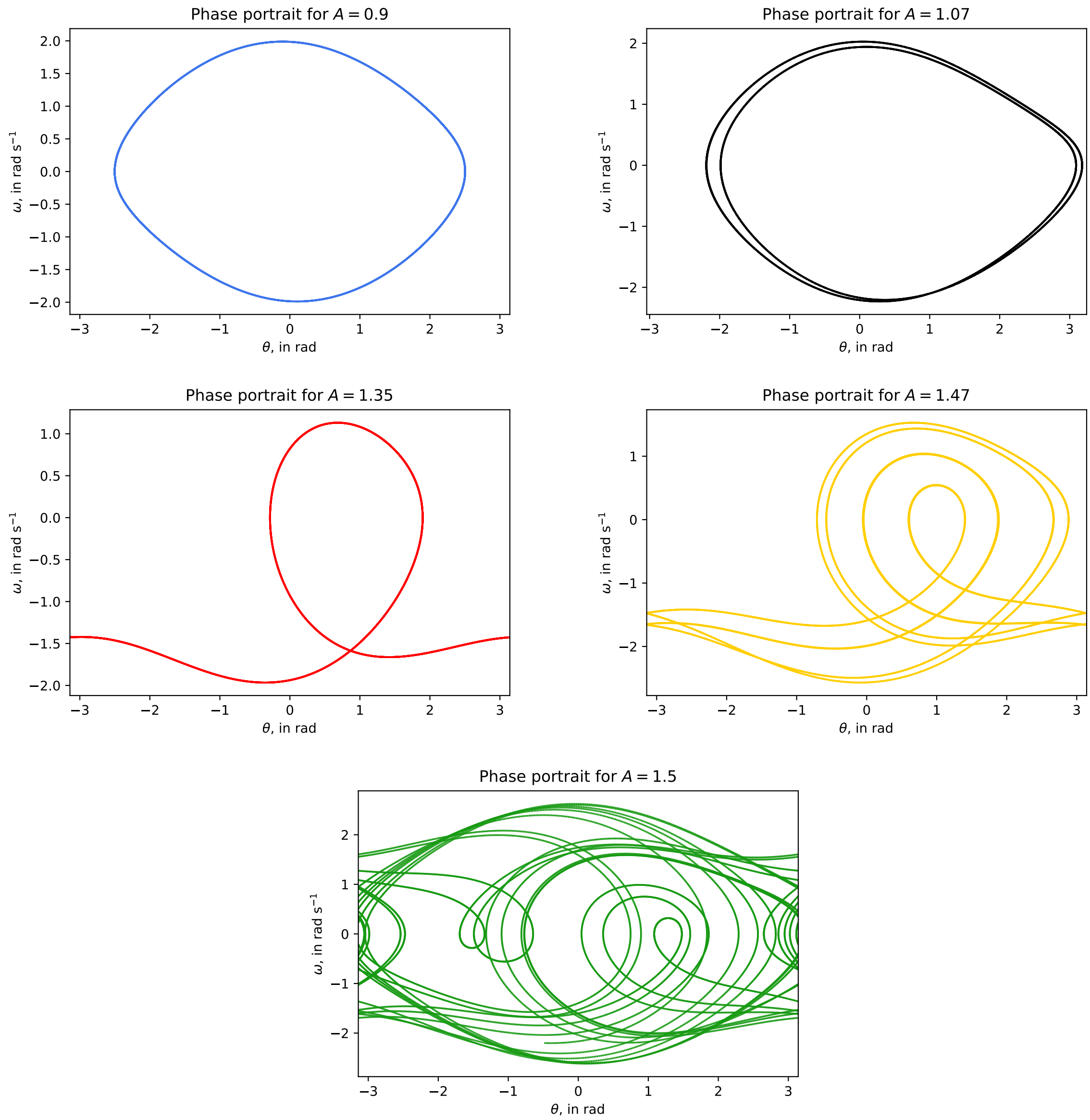

Figure 5: Phase portraits for the damped, driven pendulum for various values of A, with $\theta_0 = 1.0$, $\omega_0 = 0.0$, and time range 100-250 seconds.

4 Discussion

4.1 Linear vs Nonlinear Pendulum

The approximation of $\sin\theta \approx \theta$ is equivalent to taking the first term of the Taylor series expansion of $\sin\theta$, and is quite accurate for small values of θ. In Figure 2, the plots of $\theta(t)$ and $\omega(t)$ of the linear and nonlinear pendulum disagree only slightly for small values $\theta_0 = 0.2$, $\theta_0 = 1.0$ and $\omega_0 = 1.0$. For $\theta_0 = 3.14$, however, the graphs differ greatly. The linear motion is completely sinusoidal, whereas the nonlinear motion suggests that the pendulum pauses for $\theta \approx \pm\pi$, and swings back quickly. The latter case is more realistic, as the pendulum has an unstable equilibrium at $\theta = \pm\pi$, which means that its kinetic energy would be near a minimum and its potential near a maximum, slowing the pendulum down. This showcases the fact that the linear approximation is only appropriate for small values of θ, as $\sin(3.14) \approx 0.0016 \ll 3.14$.

4.2 Trapezoidal Rule vs Runge-Kutta Method

It can be seen from Figure 3 that the trapezoidal rule and Runge-Kutta method give largely different results, over a long enough time period. This is due to the order of time up to which each method is accurate. The plot of $\theta(t)$ using the trapezoidal rule suggests that, after approximately 40 seconds, the pendulum does not stop and swing back, but rather continues swinging in one direction. The Runge-Kutta method, however, suggests that the pendulum does indeed swing back and forth between $-\pi$ and π. The latter is clearly the more realistic scenario, which was expected due to the significant difference in highest order of time used in each method.

4.3 Damped Pendulum

The plots of the damped pendulum motion were as expected, as the amplitude of both θ and ω gradually decreased until there was very little motion ($t \approx 25$ s). Again, this highlights the accuracy of the Runge-Kutta method.

4.4 Periodic vs Chaotic Motion

Although the values of A do not vary greatly in Exercise 5, the resulting phase portraits are all quite different. For $A = 0.9$, the portrait is a normally closed loop, which shows that the motion is periodic with a constant period. This is similar for $A = 1.35$. Even though the portrait does not seem to suggest periodic motion, the loop is indeed closed, as it ‘wraps around’, corresponding to θ passing through $\pm\pi$. However, for $A = 1.07$, there seems to be two loops in the phase portrait. This is not the case, as the portrait is still one closed loop. The motion is periodic, but in fact changes between two similar periods. This is a similar case for $A = 1.47$, where the loop ‘wraps around’ as for $A = 1.35$, and switches between 4 similar periods. The only outlier is the portrait for $A = 1.5$, as this motion is completely chaotic. Nowhere in this portrait is there a closed loop; the final point of the portrait can be seen at approximately $\theta = -0.5$ rad and $\omega = -2$ rad s^{-1}. If this portrait was illustrated for a longer time period, it would simply fill up more of the phase space, never looping back on itself. This is a classic example of the so-called butterfly effect,[2] showcasing how sensitive systems can be with only a slight change of parameters, as $A = 1.47$ and $A = 1.5$ lead to completely different motion.

5 Conclusions

After plotting $\theta(t)$ and $\omega(t)$ for different pendulums using different methods, it is clear that not every approach to numerically solving differential equations yields the same results. The linear approximation $\sin\theta \approx \theta$ is only valid for small values of θ, as was seen when $\theta > 1$. Over a long time period, the trapezoidal rule also fails to be accurate for large enough θ, as was seen when compared to the higher order Runge-Kutta method for $\theta = 3.14$. The high accuracy of the Runge-Kutta method was also shown when damped motion was plotted.

It was also shown that basic systems can be very sensitive to initial conditions, where a difference of 0.03 in the driving coefficient can switch the motion from periodic to chaotic.

6 References & Appendix

1: K. E. Atkinson, *An Introduction to Numerical Analysis*, Wiley, Hoboken, 1989.

2: K. Lorenz, *Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?*, American Association for the Advancement of Science 139$^{\text{th}}$ Meeting, 1972.

All code and figures used in this laboratory can be found here:
`https://github.com/campioru/SF_Lab_2`

[2] K. Lorenz, 1972