MAU34406: Statistical Physics II
Homework 2 due 14/03/2022

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JS Theoretical Physics

Exercise 1

1.

$$\begin{aligned}\ln\Xi &= -\ln(1-z) + V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\operatorname{Li}_{\frac{5}{2}}(z)\\ &= -\ln\left(1-e^{\beta\mu}\right) + V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right)\end{aligned}$$

$$\begin{aligned}J &= -kT\ln\Xi\\ &= \beta^{-1}\ln\left(1-e^{\beta\mu}\right) - V\,(\text{deg.})\left(\frac{2\pi m}{h^2}\right)^{\frac{3}{2}}\beta^{-\frac{5}{2}}\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right)\end{aligned}$$

$$\begin{aligned}S &= -\frac{\partial J}{\partial T}\\ &= -\frac{\partial\beta}{\partial T}\frac{\partial J}{\partial\beta}\\ &= -\frac{\partial}{\partial T}\left(\frac{1}{kT}\right)\frac{\partial}{\partial\beta}\left[\beta^{-1}\ln\left(1-e^{\beta\mu}\right) - V\,(\text{deg.})\left(\frac{2\pi m}{h^2}\right)^{\frac{3}{2}}\beta^{-\frac{5}{2}}\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right)\right]\\ &= \frac{1}{kT^2}\left[-\beta^{-2}\ln\left(1-e^{\beta\mu}\right) + \frac{\beta^{-1}}{1-e^{\beta\mu}}\left(-\mu e^{\beta\mu}\right) - V\,(\text{deg.})\left(\frac{2\pi m}{h^2}\right)^{\frac{3}{2}}\left(-\frac{5\beta^{-\frac{7}{2}}}{2}\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right) + \beta^{-\frac{5}{2}}\frac{\partial e^{\beta\mu}}{\partial\beta}\frac{\partial\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right)}{\partial e^{\beta\mu}}\right)\right]\\ &= k\beta^2\left[-\beta^{-2}\ln\left(1-e^{\beta\mu}\right) - \beta^{-2}\frac{e^{\beta\mu}\beta\mu}{1-e^{\beta\mu}} - V\,(\text{deg.})\left(\frac{2\pi m}{h^2}\right)^{\frac{3}{2}}\left(-\frac{5\beta^{-\frac{7}{2}}}{2}\operatorname{Li}_{\frac{5}{2}}\left(e^{\beta\mu}\right) + \beta^{-\frac{7}{2}}e^{\beta\mu}\beta\mu\frac{\operatorname{Li}_{\frac{3}{2}}\left(e^{\beta\mu}\right)}{e^{\beta\mu}}\right)\right]\end{aligned}$$

$$S = -k\left[\ln(1-z) + \frac{z\ln(z)}{1-z} + V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\left(\ln(z)\operatorname{Li}_{\frac{3}{2}}(z) - \frac{5}{2}\operatorname{Li}_{\frac{5}{2}}(z)\right)\right]$$

In the case where $T \ll T_c$ we have $z \approx 1$.

$$\begin{aligned}\beta^{-\frac{3}{2}}\left(\ln(z)\operatorname{Li}_{\frac{3}{2}}(z) - \frac{5}{2}\operatorname{Li}_{\frac{5}{2}}(z)\right) &\approx \beta^{-\frac{3}{2}}\left(\ln(1)\operatorname{Li}_{\frac{3}{2}}(1) - \frac{5}{2}\operatorname{Li}_{\frac{5}{2}}(1)\right)\\ &\ll 1\\ &\left(\beta^{-\frac{3}{2}} \ll 1,\ \ln(1) = 0,\ \operatorname{Li}_{\frac{3}{2}}(1) = \zeta\left(\tfrac{3}{2}\right) \approx 2.61,\ \operatorname{Li}_{\frac{5}{2}}(1) = \zeta\left(\tfrac{5}{2}\right) \approx 1.34\right)\\ \lim_{z\to1}\frac{z\ln(z)}{1-z} &= \lim_{z\to1}\frac{\ln(z)+\frac{z}{z}}{-1}\\ &= -1\\ \lim_{z\to1}\ln(1-z) &= -\infty\end{aligned}$$

$$\implies S \approx -k\ln(1-z) \text{ for } T \ll T_c$$

In the case where $T \gg T_c$ we have $z \ll 1$.

$$
\begin{aligned}
-\ln(1-z) &\ll 1 \\
\mathrm{Li}_r(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n^r} \\
&\approx z \\
\implies \frac{5}{2}\,\mathrm{Li}_{\frac{5}{2}}(z) &\ll -\ln(z)\,\mathrm{Li}_{\frac{3}{2}}(z)
\end{aligned}
$$

$$
\begin{aligned}
\implies S &\approx -kz\ln(z)\left[\frac{1}{1-z} + V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}\right] \\
S &\approx -kz\ln(z)V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \qquad \left(\beta^{-\frac{3}{2}} \gg 1,\ \tfrac{1}{1-z} \approx 1\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
\langle N\rangle &= z\frac{\partial \ln \Xi}{\partial z} \\
&= \frac{z}{1-z} + V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(z)
\end{aligned}
$$

$$
\begin{aligned}
\langle E\rangle &= \mu\,\langle N\rangle - \frac{\partial \ln \Xi}{\partial \beta} \\
&= \frac{3V}{2\beta}\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{5}{2}}(z)
\end{aligned}
$$

$$
\begin{aligned}
S &= -k\left\{\ln(1-z) + \left[\frac{z}{1-z} + V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(z)\right]\ln(z) - \frac{5\beta}{3}\frac{3V}{2\beta}\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{5}{2}}(z)\right\} \\
S &= -k\left(\ln(1-z) + \langle N\rangle\ln(z) - \frac{5\beta}{3}\langle E\rangle\right)
\end{aligned}
$$

3.

$$
\begin{aligned}
N &= \langle n_0\rangle + N_{\mathrm{exc}} \\
&= \frac{z}{1-z} + V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(z) \\
\implies \langle n_0\rangle &= \frac{z}{1-z} \text{ and } N_{\mathrm{exc}} = V\,(\mathrm{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(z)
\end{aligned}
$$

In the case where $T \ll T_c$ we have $\langle n_0\rangle \approx N$.

$$
\begin{aligned}
\langle n_0\rangle &= \frac{z}{1-z} \\
\implies z &= \frac{1}{1+\frac{1}{\langle n_0\rangle}} \\
&\approx \frac{1}{1+\frac{1}{N}} \\
&= 1 - \frac{1}{N} + \mathcal{O}(N^{-2}) \\
\implies S &\approx -k\ln\left(1 - 1 + \frac{1}{N}\right) \\
S &\approx k\ln(N)
\end{aligned}
$$

In the case where $T \ll T_c$ we have $z \ll 1$, $N_{\text{exc}} \approx N$, and $\langle n_0 \rangle \ll 1$.

$$\begin{aligned} N_{\text{exc}} &= V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \operatorname{Li}_{\frac{3}{2}}(z) \\ \implies N &\approx V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} z \\ \implies S &\approx -kN\ln(z) \end{aligned}$$

$$\begin{aligned} \langle n_0 \rangle &= N - N_{\text{exc}} \\ &= N - V\,(\text{deg.})\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}} \operatorname{Li}_{\frac{3}{2}}(z) \\ &= N - \left[\frac{h^2}{2\pi m k}\left(\frac{N}{V\,(\text{deg.})\operatorname{Li}_{\frac{3}{2}}(1)}\right)^{\frac{2}{3}}\right]^{-\frac{3}{2}} N T^{\frac{3}{2}} \frac{\operatorname{Li}_{\frac{3}{2}}(z)}{\operatorname{Li}_{\frac{3}{2}}(1)} \\ &\approx N\left[1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}} \frac{z}{\zeta(\frac{3}{2})}\right] \\ &\ll 1 \\ \implies z &\approx \zeta\left(\frac{3}{2}\right)\left(\frac{T_c}{T}\right)^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \implies S &\approx -kN\ln\left[\zeta\left(\frac{3}{2}\right)\left(\frac{T_c}{T}\right)^{\frac{3}{2}}\right] \\ &= kN\left[\frac{3}{2}\ln\left(\frac{T}{T_c}\right) - \ln\zeta\left(\frac{3}{2}\right)\right] \\ S &\approx kN\left[\frac{3}{2}\ln\left(\frac{T}{T_c}\right)\right] \qquad (T \gg T_c) \end{aligned}$$

Exercise 2

$$\begin{aligned}
P &\approx \frac{2\langle E\rangle}{3V} \\
&= \frac{3}{2}\left(\frac{2\pi mk}{h^2}\right)^{\frac{3}{2}} T^{\frac{5}{2}} \operatorname{Li}_{\frac{5}{2}}(z) \\
0 &= \left(\frac{\partial P}{\partial T}\right)_P \\
&= \frac{3}{2}\left(\frac{2\pi mk}{h^2}\right)^{\frac{3}{2}} \left[\frac{5T^{\frac{3}{2}}}{2}\operatorname{Li}_{\frac{5}{2}}(z) + \frac{T^{\frac{5}{2}}}{z} z \left(\frac{\partial \operatorname{Li}_{\frac{5}{2}}(z)}{\partial z}\right)_P \left(\frac{\partial z}{\partial T}\right)_P\right] \\
\implies 0 &= \frac{5T^{\frac{3}{2}}}{2}\operatorname{Li}_{\frac{5}{2}}(z) + T^{\frac{5}{2}}\operatorname{Li}_{\frac{3}{2}}(z)\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_P \\
\implies \frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_P &= -\frac{5}{2T}\frac{\operatorname{Li}_{\frac{5}{2}}(z)}{\operatorname{Li}_{\frac{3}{2}}(z)}
\end{aligned}$$

$$\begin{aligned}
N &= \langle N\rangle \\
&\approx V\,(\text{deg.})\left(\frac{2\pi mk}{h^2}\right)^{\frac{3}{2}} T^{\frac{3}{2}}\operatorname{Li}_{\frac{3}{2}}(z) \qquad (T \gg T_c \implies z \ll 1) \\
0 &= \left(\frac{\partial N}{\partial T}\right)_N \\
&= \left(\frac{\partial N}{\partial T}\right)_v \\
&= V\,(\text{deg.})\left(\frac{2\pi mk}{h^2}\right)^{\frac{3}{2}} \left[\frac{3T^{\frac{1}{2}}}{2}\operatorname{Li}_{\frac{3}{2}}(z) + \frac{T^{\frac{3}{2}}}{z} z\left(\frac{\partial \operatorname{Li}_{\frac{3}{2}}(z)}{\partial z}\right)_v \left(\frac{\partial z}{\partial T}\right)_v\right] \\
\implies 0 &= \frac{3T^{\frac{1}{2}}}{2}\operatorname{Li}_{\frac{3}{2}}(z) + T^{\frac{3}{2}}\operatorname{Li}_{\frac{1}{2}}(z)\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v \\
\implies \frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v &= -\frac{3}{2T}\frac{\operatorname{Li}_{\frac{3}{2}}(z)}{\operatorname{Li}_{\frac{1}{2}}(z)}
\end{aligned}$$

$$\begin{aligned}
C_P &= T\left(\frac{\partial S}{\partial T}\right)_{N,P} & C_V &= T\left(\frac{\partial S}{\partial T}\right)_{N,V} \\
&= T\left(\frac{\partial S}{\partial z}\right)_N \left(\frac{\partial z}{\partial T}\right)_P & &= T\left(\frac{\partial S}{\partial z}\right)_N \left(\frac{\partial z}{\partial T}\right)_v
\end{aligned}$$

$$\begin{aligned}
\gamma &= \frac{C_P}{C_V} \\
&= \frac{T\left(\frac{\partial S}{\partial z}\right)_N\left(\frac{\partial z}{\partial T}\right)_P}{T\left(\frac{\partial S}{\partial z}\right)_N\left(\frac{\partial z}{\partial T}\right)_v} \\
&= \frac{\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_P}{\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v} \\
&= \frac{5}{3}\frac{\operatorname{Li}_{\frac{5}{2}}(z)\operatorname{Li}_{\frac{1}{2}}(z)}{\left(\operatorname{Li}_{\frac{3}{2}}(z)\right)^2}
\end{aligned}$$

Exercise 3

$$
\begin{aligned}
\ln \Xi &= -\ln(1-z) - (\text{deg.})\,\frac{V_d}{h^d}\frac{(2m)^{\frac{d}{2}}}{2}\int_0^\infty \epsilon^{\frac{d}{2}-1}\ln\!\left(1 - ze^{-\beta\epsilon}\right) d\epsilon \\
N_{\text{exc}} &= z\frac{\partial \ln \Xi}{\partial z} \\
&= 0 - z\,(\text{deg.})\,\frac{V_d}{h^d}\frac{(2m)^{\frac{d}{2}}}{2}\int_0^\infty \epsilon^{\frac{d}{2}-1}\frac{-ze^{-\beta\epsilon}}{1-ze^{-\beta\epsilon}}\,d\epsilon \\
&= (\text{deg.})\,\frac{V_d}{h^d}\frac{(2m)^{\frac{d}{2}}}{2}\int_0^\infty \epsilon^{\frac{d}{2}-1}\frac{ze^{-\beta\epsilon}}{1-ze^{\beta\epsilon}}\,d\epsilon \\
&= (\text{deg.})\,\frac{V_d}{h^d}\frac{(2m)^{\frac{d}{2}}}{2}\int_0^\infty d(\beta\epsilon)\,\frac{(\beta\epsilon)^{\frac{d}{2}-1}}{z^{-1}e^{\beta\epsilon}-1}\,\beta^{-\frac{d}{2}} \\
&= (\text{deg.})\,\frac{V_d}{h^d}\frac{(2m)^{\frac{d}{2}}}{2}\beta^{-\frac{d}{2}}\Gamma\!\left(\frac{d}{2}\right)\mathrm{Li}_{\frac{d}{2}}(z)
\end{aligned}
$$

For Bose-Einstein condensation to occur we need an upper bound on the number of particles occupying excited states when $z \approx 1$. However, for $z \approx 1$ we have $N_{\text{exc}} \propto \mathrm{Li}_{\frac{d}{2}}(1) = \zeta\left(\frac{d}{2}\right)$, which does not converge for $d = 1, 2$. Thus, Bose-Einstein condensation cannot occur for $d = 1, 2$.

Exercise 4

$$
\begin{aligned}
\mathcal{W}(1,p)\,dp &= (d-2)\frac{VS_{d-2}}{h^{d-1}}p^{d-2}\,dp \\
&= (d-2)\frac{VS_{d-2}}{(hc)^{d-1}}\epsilon^{d-2}\,d\epsilon && (\epsilon = cp) \\
&= (d-2)\frac{VS_{d-2}}{(hc)^{d-1}}(\hbar\omega)^{d-2}\,d(\hbar\omega) && (\epsilon = \hbar\omega) \\
\implies \mathcal{W}(1,\omega)\,d\omega &= (d-2)\frac{VS_{d-2}}{(2\pi c)^{d-1}}\omega^{d-2}\,d\omega
\end{aligned}
$$

$$
\begin{aligned}
E(\epsilon)\,d\epsilon &= \langle n(\epsilon)\rangle\,\epsilon\mathcal{W}(1,\epsilon)\,d\epsilon \\
&= \frac{d-2}{e^{\beta\epsilon}-1}\frac{VS_{d-2}}{(hc)^{d-1}}\epsilon^{d-1}\,d\epsilon \\
&= \frac{d-2}{e^{\beta\hbar\omega}-1}\frac{VS_{d-2}}{(hc)^{d-1}}(\hbar\omega)^{d-1}\hbar\,d\omega && (\epsilon = \hbar\omega) \\
&= \frac{d-2}{e^{\beta\hbar\omega}-1}\frac{\hbar VS_{d-2}}{(2\pi c)^{d-1}}\omega^{d-1}\,d\omega
\end{aligned}
$$

$$
\begin{aligned}
\langle E \rangle &= \int_0^\infty E(\epsilon)\, d\epsilon \\
&= \frac{\hbar(d-2)VS_{d-2}}{(2\pi c)^{d-1}} \int_0^\infty \frac{\omega^{d-1}}{e^{\beta\hbar\omega}-1}\, d\omega \\
&= \frac{\hbar(d-2)VS_{d-2}}{(2\pi c)^{d-1}} \int_0^\infty (\beta\hbar)^{-d} \frac{(\beta\hbar\omega)^{d-1}}{e^{\beta\hbar\omega}-1}\, d(\beta\hbar\omega) \\
&= \frac{\beta^{-d}(d-2)VS_{d-2}}{(2\pi\hbar c)^{d-1}} \Gamma(d)\, \mathrm{Li}_d(1) \\
&= \frac{k^d(d-2)VS_{d-2}}{(hc)^{d-1}} \Gamma(d)\zeta(d)T^d \\
\langle \epsilon \rangle &= \frac{\langle E \rangle}{V} \\
\langle \epsilon \rangle &= \frac{k^d(d-2)S_{d-2}}{(hc)^{d-1}} \Gamma(d)\zeta(d)T^d
\end{aligned}
$$

$$
\begin{aligned}
-PV &= J \\
P &= -\frac{J}{V} \\
&= \frac{kT}{V} \ln \Xi \\
&= -\frac{1}{\beta V} \int_0^\infty \mathcal{W}(1,\omega) \ln\left(1-e^{-\beta\hbar\omega}\right) d\omega \\
&= -\frac{d-2}{\beta} \frac{S_{d-2}}{(2\pi c)^{d-1}} \int_0^\infty \omega^{d-2} \ln\left(1-e^{-\beta\hbar\omega}\right) d\omega \\
&= -\frac{d-2}{\beta} \frac{S_{d-2}}{(2\pi c)^{d-1}} \left(\left. \frac{\omega^{d-1} \ln\left(1-e^{-\beta\hbar\omega}\right)}{d-1} \right|_0^\infty - \int_0^\infty \frac{\omega^{d-1}}{(d-1)\left(1-e^{-\beta\hbar\omega}\right)} \left(-e^{-\beta\hbar\omega}(-\beta\hbar)\right) \right) \\
&\qquad \left(u = \ln\left(1-e^{-\beta\hbar\omega}\right),\ dv = \omega^{d-2}\, d\omega\right) \\
&= \frac{1}{d-1} \frac{\hbar(d-2)S_{d-2}}{(2\pi c)^{d-1}} \int_0^\infty \frac{\omega^{d-1}}{e^{\beta\hbar\omega}-1}\, d\omega \\
&= \frac{1}{d-1} \frac{\langle E \rangle}{V} \\
P &= \frac{\langle \epsilon \rangle}{d-1}
\end{aligned}
$$