MAU34403: Quantum Mechanics I
Homework 8 due 19/11/2021

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JS Theoretical Physics

Problem 1.

(a)

Assume $\psi(x)$ takes the form

$$\psi(x) = \left\{ \begin{array}{ll} \psi_L(x) = A_L\, e^{i\,k_L\,x} + B_L\, e^{-i\,k_L\,x}, & x \leqslant -a \\ \psi_R(x) = A_R\, e^{i\,k_R\,x} + B_R\, e^{-i\,k_R\,x}, & x \geqslant a \end{array} \right. ,$$
$$\text{where } k_\alpha \equiv \frac{\sqrt{2m\,(E - V_\alpha)}}{\hbar},\ \alpha = L, R.$$

$$V_L = V_R = 0 \implies k_L = k_R$$
$$\implies \psi(x) = \left\{ \begin{array}{ll} \psi_L(x) = A_L\, e^{i\,k\,x} + B_L\, e^{-i\,k\,x}, & x \leqslant -a \\ \psi_R(x) = A_R\, e^{i\,k\,x} + B_R\, e^{-i\,k\,x}, & x \geqslant a \end{array} \right. ,\ \text{where } k \equiv \frac{\sqrt{2m\,E}}{\hbar}$$

$$\psi(x) \text{ is continuous at } a \implies \psi_L(a) = \psi_R(a)$$
$$\implies A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} = A_R\, e^{i\,k\,a} + B_R\, e^{-i\,k\,a} \tag{1}$$

$$\begin{aligned}
V(x)\,\psi(x) &= \frac{\hbar^2}{2m}\,\psi''(x) + E\,\psi(x) \\
\lim_{\varepsilon\to 0}\int_{-\varepsilon+a}^{\varepsilon+a} V(x)\,\psi(x)\,dx &= \lim_{\varepsilon\to 0}\int_{-\varepsilon+a}^{\varepsilon+a}\left(\frac{\hbar^2}{2m}\,\psi''(x) + E\,\psi(x)\right)dx \\
\lim_{\varepsilon\to 0}\int_{-\varepsilon+a}^{\varepsilon+a} \nu\,\delta(x)\,\psi(x)\,dx &= \frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\int_{-\varepsilon+a}^{\varepsilon+a}\psi''(x)\,dx + E\lim_{\varepsilon\to 0}\int_{-\varepsilon+a}^{\varepsilon+a}\psi(x)\,dx \\
\lim_{\varepsilon\to 0}\nu\,\psi(a) &= \frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}\psi'(x)\Big|_{-\varepsilon+a}^{\varepsilon+a} + 0 \qquad \text{(since } \psi(x) \text{ is continuous at } x = a\text{)} \\
\psi(a) &= \frac{\hbar^2}{2m\,\nu}\left(\psi_R'(a) - \psi_L'(a)\right) \\
A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} &= \frac{\hbar^2}{2m\,\nu}\left(i\,k\,A_R\, e^{i\,k\,a} - i\,k\,B_R\, e^{-i\,k\,a} - i\,k\,A_L\, e^{i\,k\,a} + i\,k\,B_L\, e^{-i\,k\,a}\right) \\
&\qquad (\psi_L(a) = \psi_R(a) = \psi(a)) \\
&= \frac{i\,k\,\hbar^2}{2m\,\nu}\left(A_R\, e^{i\,k\,a} - B_R\, e^{-i\,k\,a} - A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a}\right)
\end{aligned}$$

$$\begin{aligned}
(1) \implies B_R\, e^{-i\,k\,a} &= A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} - A_R\, e^{i\,k\,a} \\
\implies A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} &= \frac{i\,k\,\hbar^2}{2m\,\nu}\left(A_R\, e^{i\,k\,a} - A_L\, e^{i\,k\,a} - B_L\, e^{-i\,k\,a} + A_R\, e^{i\,k\,a} - A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a}\right) \\
&= \frac{i\,k\,\hbar^2}{m\,\nu}\left(A_R\, e^{i\,k\,a} - A_L\, e^{i\,k\,a}\right) \\
A_R\,\frac{i\,k\,\hbar^2}{m\,\nu}\, e^{i\,k\,a} &= A_L\left(1 + \frac{i\,k\,\hbar^2}{m\,\nu}\right) e^{i\,k\,a} + B_L\, e^{-i\,k\,a} \\
A_R &= A_L\left(\frac{m\,\nu}{i\,k\,\hbar^2} + 1\right) + B_L\,\frac{m\,\nu}{i\,k\,\hbar^2}\, e^{-2i\,k\,a} \\
&= A_L\left(1 - \frac{i\,m\,\nu}{k\,\hbar^2}\right) - B_L\,\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a}
\end{aligned} \tag{2}$$

$$
\begin{aligned}
(1) \implies A_R\, e^{-i\,k\,a} &= A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} - B_R\, e^{i\,k\,a} \\
\implies A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} &= \frac{i\,k\,\hbar^2}{2m\,\nu}\left(A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a} - B_R\, e^{i\,k\,a} - B_R\, e^{-i\,k\,a} - A_L\, e^{i\,k\,a} + B_L\, e^{-i\,k\,a}\right) \\
&= \frac{i\,k\,\hbar^2}{m\,\nu}\left(B_L\, e^{-i\,k\,a} - B_R\, e^{-i\,k\,a}\right) \\
B_R\,\frac{i\,k\,\hbar^2}{m\,\nu}\, e^{-i\,k\,a} &= -A_L\, e^{i\,k\,a} + B_L\left(\frac{i\,k\,\hbar^2}{m\,\nu} - 1\right) e^{-i\,k\,a} \\
B_R &= -A_L\,\frac{m\,\nu}{i\,k\,\hbar^2}\, e^{2i\,k\,a} + B_L\left(1 - \frac{m\,\nu}{i\,k\,\hbar^2}\right) \\
&= A_L\,\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} + B_L\left(1 + \frac{i\,m\,\nu}{k\,\hbar^2}\right) \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
(2),(3) \implies \begin{pmatrix} A_R \\ B_R \end{pmatrix} &= \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix} \\
\implies M &= \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
M_1 &= M|_{a\to -a} \\
&= \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix}
\end{aligned}
\qquad
\begin{aligned}
M_2 &= M|_{a\to a} \\
&= \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M &= M_2\, M_1 \\
&= \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix} \begin{pmatrix} 1 - \frac{i\,m\,\nu}{k\,\hbar^2} & -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} \\ \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} & 1 + \frac{i\,m\,\nu}{k\,\hbar^2} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{AA} &= 1 - \frac{2i\,m\,\nu}{k\,\hbar^2} + \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 - \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{-4i\,k\,a} \\
&= \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{-4i\,k\,a} - 1\right) - \frac{2i\,m\,\nu}{k\,\hbar^2} + 1
\end{aligned}
$$

$$
\begin{aligned}
M_{AB} &= -\frac{i\,m\,\nu}{k\,\hbar^2}\, e^{2i\,k\,a} + \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{2i\,k\,a} - \frac{i\,m\,\nu}{k\,\hbar^2}\, e^{-2i\,k\,a} - \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{-2i\,k\,a} \\
&= \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{-2i\,k\,a} - e^{2i\,k\,a}\right) - \frac{i\,m\,\nu}{k\,\hbar^2}\left(e^{2i\,k\,a} + e^{-2i\,k\,a}\right) \\
&= -2i\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \sin(2k\,a) - \frac{2i\,m\,\nu}{k\,\hbar^2}\cos(2k\,a) \\
&= -\frac{2i\,m\,\nu}{k\,\hbar^2}\left(\frac{m\,\nu}{k\,\hbar^2}\sin(2k\,a) + \cos(2k\,a)\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{BA} &= \frac{i\,m\,\nu}{k\,\hbar^2}\,e^{2i\,k\,a} - \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{2i\,k\,a} + \frac{i\,m\,\nu}{k\,\hbar^2}\,e^{-2i\,k\,a} + \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{-2i\,k\,a} \\
&= \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{2i\,k\,a} - e^{-2i\,k\,a}\right) + \frac{i\,m\,\nu}{k\,\hbar^2}\left(e^{2i\,k\,a} + e^{-2i\,k\,a}\right) \\
&= 2i\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \sin(2k\,a) + \frac{2i\,m\,\nu}{k\,\hbar^2}\cos(2k\,a) \\
&= \frac{2i\,m\,\nu}{k\,\hbar^2}\left(\frac{m\,\nu}{k\,\hbar^2}\sin(2k\,a) + \cos(2k\,a)\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{BB} &= -\left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 e^{4i\,k\,a} + 1 + \frac{2i\,m\,\nu}{k\,\hbar^2} + \left(\frac{i\,m\,\nu}{k\,\hbar^2}\right)^2 \\
&= \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{4i\,k\,a} - 1\right) + \frac{2i\,m\,\nu}{k\,\hbar^2} + 1
\end{aligned}
$$

$$
\implies M = \begin{pmatrix} \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{-4i\,k\,a} - 1\right) - \frac{2i\,m\,\nu}{k\,\hbar^2} + 1 & -\frac{2i\,m\,\nu}{k\,\hbar^2}\left(\frac{m\,\nu}{k\,\hbar^2}\sin(2k\,a) + \cos(2k\,a)\right) \\ \frac{2i\,m\,\nu}{k\,\hbar^2}\left(\frac{m\,\nu}{k\,\hbar^2}\sin(2k\,a) + \cos(2k\,a)\right) & \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{4i\,k\,a} - 1\right) + \frac{2i\,m\,\nu}{k\,\hbar^2} + 1 \end{pmatrix}
$$

$$
\begin{aligned}
B_R &= M_{BA}\,A_L + M_{BB}\,B_L \\
\implies B_L &= -\frac{M_{BA}}{M_{BB}}\,A_L + \frac{1}{M_{BB}}\,B_R \\
&= S_{LL}\,A_L + S_{LR}\,B_R \\
\implies S_{LR} &= \frac{1}{M_{BB}}
\end{aligned}
$$

$$
\begin{aligned}
|M_{BB}|^2 &= \left|\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \left(e^{4i\,k\,a} - 1\right) + \frac{2i\,m\,\nu}{k\,\hbar^2} + 1\right|^2 \\
&= \left|\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 (\cos(4k\,a) + i\sin(4k\,a)) - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + \frac{2i\,m\,\nu}{k\,\hbar^2} + 1\right|^2 \\
&= \left|\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \cos(4k\,a) - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + 1 + i\left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \sin(4k\,a) + \frac{2m\,\nu}{k\,\hbar^2}\right)\right|^2 \\
&= \left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \cos(4k\,a) - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + 1\right)^2 + \left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \sin(4k\,a) + \frac{2m\,\nu}{k\,\hbar^2}\right)^2 \\
&= 2\left(\frac{m\,\nu}{k\,\hbar^2}\right)^4 + 2\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + 1 + 2\left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^4\right)\cos(4a\,k) + 4\left(\frac{m\,\nu}{k\,\hbar^2}\right)^3 \sin(4a\,k)
\end{aligned}
$$

$$
\left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \mathrm{Cos}[4\,k\,a] - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + 1\right)^2 + \left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 \mathrm{Sin}[4\,k\,a] + \frac{2\,m\,\nu}{k\,\hbar^2}\right)^2 \;\text{// Expand // FullSimplify}
$$

$$
\frac{2\,m^4\,\nu^4 + 2\,k^2\,m^2\,\nu^2\,\hbar^4 + k^4\,\hbar^8 + 2\,m^2\,\nu^2\left(\left(-m^2\,\nu^2 + k^2\,\hbar^4\right)\mathrm{Cos}[4\,a\,k] + 2\,k\,m\,\nu\,\hbar^2\,\mathrm{Sin}[4\,a\,k]\right)}{k^4\,\hbar^8}
$$

$$
\begin{aligned}
T &= |S_{LR}|^2 \\
&= \frac{1}{|M_{BB}|^2} \\
&= \frac{1}{2\left(\frac{m\,\nu}{k\,\hbar^2}\right)^4 + 2\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 + 1 + 2\left(\left(\frac{m\,\nu}{k\,\hbar^2}\right)^2 - \left(\frac{m\,\nu}{k\,\hbar^2}\right)^4\right)\cos(4a\,k) + 4\left(\frac{m\,\nu}{k\,\hbar^2}\right)^3 \sin(4a\,k)}
\end{aligned}
$$

Problem 2

(a)

Define centre of mass position $\bar{X}$ and momentum $\bar{P}$ and relative position X and momentum P as

$$\begin{aligned}\bar{X} &= \frac{m\,X_1 + m\,X_2}{m+m} \\ &= \frac{1}{2}\,(X_1 + X_2) \\ \bar{P} &= P_1 + P_2\end{aligned} \qquad \begin{aligned} X &= X_1 - X_2 \\ P &= \frac{m\,P_1 - m\,P_2}{m+m} \\ &= \frac{1}{2}\,(P_1 - P_2)\end{aligned}$$

We can rearrange these expressions to find

$$\begin{aligned} X_1 &= \bar{X} + \frac{1}{2}\,X \\ P_1 &= \frac{1}{2}\,\bar{P} + P \end{aligned} \qquad \begin{aligned} X_2 &= \bar{X} - \frac{1}{2}\,X \\ P_2 &= \frac{1}{2}\,\bar{P} - P \end{aligned}$$

$$\begin{aligned}
H\big(\bar{X},\bar{P},X,P\big) &= H\big(X_1\big(\bar{X},X\big)\,, X_2\big(\bar{X},X\big)\,, P_1\big(\bar{P},P\big)\,, P_2\big(\bar{P},P\big)\big) \\
&= \frac{\left(\frac{1}{2}\,\bar{P}+P\right)^2}{2m} + \frac{\left(\frac{1}{2}\,\bar{P}-P\right)^2}{2m} + \frac{k\,\left(\bar{X}+\frac{1}{2}\,X\right)^2}{2} + \frac{k\,\left(\bar{X}-\frac{1}{2}\,X\right)^2}{2} + V\left(\bar{X}+\frac{1}{2}\,X - \bar{X} + \frac{1}{2}\,X\right) \\
&= \frac{\bar{P}^2}{4m} + \frac{P^2}{m} + k\,\bar{X}^2 + \frac{k\,X^2}{4} + V(X) \\
&= \frac{\bar{P}^2}{2\,(2m)} + \frac{(2k)\,\bar{X}^2}{2} + \frac{P^2}{2\left(\frac{m}{2}\right)} + \frac{\left(\frac{k}{2}+k_{12}\right)X^2}{2} \qquad \text{(when } X_1 > X_2\text{)} \\
&= H_{\text{CoM}}\big(\bar{X},\bar{P}\big) + H_{\text{rel}}(X,P)
\end{aligned}$$

$$\begin{aligned}
E\,\psi(\vec{x}) &= -\frac{\hbar^2}{2m}\,\nabla^2\psi(\vec{x}) + V(\vec{x},t) \\
E\,\psi(\bar{x},x) &= \left(-\frac{\hbar^2}{2\,(2m)}\,\frac{\partial^2}{\partial\bar{x}^2} - \frac{\hbar^2}{2\left(\frac{m}{2}\right)}\,\frac{\partial^2}{\partial x^2} + \frac{(2k)\,\bar{x}^2}{2} + \frac{\left(\frac{k}{2}+k_{12}\right)x^2}{2}\right)\psi(\bar{x},x) \\
E\,\varphi_{\text{CoM}}(\bar{x})\,\varphi_{\text{rel}}(x) &= \left(-\frac{\hbar^2}{2\,(2m)}\,\frac{\partial^2}{\partial\bar{x}^2} + \frac{(2k)\,\bar{x}^2}{2} - \frac{\hbar^2}{2\left(\frac{m}{2}\right)}\,\frac{\partial^2}{\partial x^2} + \frac{\left(\frac{k}{2}+k_{12}\right)x^2}{2}\right)\varphi_{\text{CoM}}(\bar{x})\,\varphi_{\text{rel}}(x) \\
&\qquad \text{(assuming } \psi \text{ is separable)} \\
E &= \frac{1}{\varphi_{\text{CoM}}(\bar{x})}\left(-\frac{\hbar^2}{2\,(2m)}\,\frac{\partial^2}{\partial\bar{x}^2} + \frac{(2k)\,\bar{x}^2}{2}\right)\varphi_{\text{CoM}}(\bar{x}) \\
&\quad + \frac{1}{\varphi_{\text{rel}}(x)}\left(-\frac{\hbar^2}{2\left(\frac{m}{2}\right)}\,\frac{\partial^2}{\partial x^2} + \frac{\left(\frac{k}{2}+k_{12}\right)x^2}{2}\right)\varphi_{\text{rel}}(x)
\end{aligned}$$

This can be separated into

$$\begin{aligned}
E_{\text{CoM}} &= \frac{1}{\varphi_{\text{CoM}}(\bar{x})}\left(-\frac{\hbar^2}{2\,(2m)}\,\frac{\partial^2}{\partial\bar{x}^2} + \frac{(2k)\,\bar{x}^2}{2}\right)\varphi_{\text{CoM}}(\bar{x}) \\
E_{\text{rel}} &= \frac{1}{\varphi_{\text{rel}}(x)}\left(-\frac{\hbar^2}{2\left(\frac{m}{2}\right)}\,\frac{\partial^2}{\partial x^2} + \frac{\left(\frac{k}{2}+k_{12}\right)x^2}{2}\right)\varphi_{\text{rel}}(x) \\
E &= E_{\text{CoM}} + E_{\text{rel}}
\end{aligned}$$

This are energy equations for harmonic oscillators, and so the respective energies can be written as

$$E_{\text{CoM}} = \hbar\sqrt{\frac{2k}{2m}}\left(n_1+\frac{1}{2}\right) \qquad E_{\text{rel}} = \hbar\sqrt{\frac{\frac{k}{2}+k_{12}}{\frac{m}{2}}}\left(n_2+\frac{1}{2}\right)$$
$$= \hbar\sqrt{\frac{k}{m}}\left(n_1+\frac{1}{2}\right) \qquad = \hbar\sqrt{\frac{k+2k_{12}}{m}}\left(n_2+\frac{1}{2}\right)$$

Since there is no restriction on E_{CoM}, there is no restriction on the typical values n_1 can take, i.e. $n_1 = 0, 1, 2, \ldots$. Since $\varphi_{\text{rel}}(0) = 0$ and the only wavefunctions of the harmonic oscillator that vanish at $x = 0$ are odd, then $\varphi_{\text{rel}}(x)$ is odd, and so $n_2 = 1, 3, 5, \ldots$. Thus the total energy is given by

$$E_{n_1,n_2} = \frac{\hbar}{\sqrt{m}}\left(\sqrt{k}\left(n_1+\frac{1}{2}\right)+\sqrt{k+2k_{12}}\left(n_2+\frac{1}{2}\right)\right),\ n_1 = 0,1,2,\ldots,\ n_2 = 1,3,5,\ldots$$

The wavefunctions φ follow the standard solutions to the harmonic potential, namely

$$\varphi_{\text{CoM}}(\bar{x}) = \frac{1}{\sqrt{2^{n_1}\,n_1!}}\,H_{n_1}\left(\frac{\bar{x}}{\eta_{\text{CoM}}\sqrt{2}}\right)\frac{1}{\sqrt{\eta_{\text{CoM}}\sqrt{2\pi}}}\exp\left(-\frac{\bar{x}^2}{4\eta_{\text{CoM}}^2}\right),$$
$$\text{where } \eta_{\text{CoM}} = \sqrt{\frac{\hbar}{2m_{\text{CoM}}\,\omega_{\text{CoM}}}} = \sqrt{\frac{\hbar}{2\,(2m)}\sqrt{\frac{2m}{2k}}} = \sqrt{\frac{\hbar}{4\sqrt{k\,m}}}$$

and

$$\varphi_{\text{rel}}(\bar{x}) = \frac{1}{\sqrt{2^{n_2}\,n_2!}}\,H_{n_2}\left(\frac{x}{\eta_{\text{rel}}\sqrt{2}}\right)\frac{1}{\sqrt{\eta_{\text{CoM}}\sqrt{2\pi}}}\exp\left(-\frac{\bar{x}^2}{4\eta_{\text{rel}}^2}\right),$$
$$\text{where } \eta_{\text{rel}} = \sqrt{\frac{\hbar}{2m_{\text{rel}}\,\omega_{\text{rel}}}} = \sqrt{\frac{\hbar}{2\left(\frac{m}{2}\right)}\sqrt{\frac{\frac{m}{2}}{\frac{k}{2}+k_{12}}}} = \sqrt{\frac{\hbar}{\sqrt{m\,(k+2k_{12})}}}$$

The total wavefunction ψ is given by the product of the above wavefunctions φ. However, since $\varphi_{\text{rel}}(x)$ is normalised for $x \in (-\infty, \infty)$ but only non-zero for $x \in [0, \infty)$, the normalised total wavefunction $\psi = \varphi_{\text{CoM}}\,\varphi_{\text{rel}}$ will require a factor of $\sqrt{2}$, and thus the normalised wavefunction for the entire system is given by

$$\psi(\bar{x}, x) = \sqrt{2}\,\frac{1}{\sqrt{2^{n_1}\,n_1!}}\,H_{n_1}\left(\frac{\bar{x}}{\eta_{\text{CoM}}\sqrt{2}}\right)\frac{1}{\sqrt{\eta_{\text{CoM}}\sqrt{2\pi}}}\exp\left(-\frac{\bar{x}^2}{4\eta_{\text{CoM}}^2}\right)\frac{1}{\sqrt{2^{n_2}\,n_2!}}\,H_{n_2}\left(\frac{x}{\eta_{\text{rel}}\sqrt{2}}\right)\frac{1}{\sqrt{\eta_{\text{CoM}}\sqrt{2\pi}}}\exp\left(-\frac{\bar{x}^2}{4\eta_{\text{rel}}^2}\right)$$
$$= \sqrt{\frac{1}{\pi\,2^{n_1+n_2}\,n_1!\,n_2!\,\eta_{CoM}\,\eta_{\text{rel}}}}\,H_{n_1}\left(\frac{\bar{x}}{\eta_{\text{CoM}}\sqrt{2}}\right)H_{n_2}\left(\frac{x}{\eta_{\text{rel}}\sqrt{2}}\right)\exp\left(-\frac{1}{4}\left(\frac{\bar{x}^2}{\eta_{\text{CoM}}^2}+\frac{x^2}{\eta_{\text{rel}}^2}\right)\right),$$

where the relevant quantities are given by

$$\bar{x} = \frac{1}{2}(x_1+x_2) \qquad x = x_1 - x_2$$
$$n_1 = 0,1,2,\ldots \qquad n_2 = 1,3,5,\ldots$$
$$\eta_{\text{CoM}} = \sqrt{\frac{\hbar}{4\sqrt{k\,m}}} \qquad \eta_{\text{rel}} = \sqrt{\frac{\hbar}{\sqrt{m\,(k+2k_{12})}}}$$

(b)

If the spectrum is degenerate then there must be some combination of n_1, n_2 that gives the same eigenvalue as n_1', n_2', for either $n_1 \neq n_1'$ or $n_2 \neq n_2'$.

$$E = E'$$

$$\frac{\hbar}{\sqrt{m}}\left(\sqrt{k}\left(n_1+\frac{1}{2}\right)+\sqrt{k+2k_{12}}\left(n_2+\frac{1}{2}\right)\right) = \frac{\hbar}{\sqrt{m}}\left(\sqrt{k}\left(n_1'+\frac{1}{2}\right)+\sqrt{k+2k_{12}}\left(n_2'+\frac{1}{2}\right)\right)$$

$$n_1\sqrt{k}+n_2\sqrt{k+2k_{12}} = n_1'\sqrt{k}+n_2'\sqrt{k+2k_{12}}$$

$$\alpha\sqrt{k}+(2\beta+1)\sqrt{k+2k_{12}} = \gamma\sqrt{k}+(2\delta+1)\sqrt{k+2k_{12}},\ \alpha,\beta,\gamma,\delta = 0,1,2,\ldots$$

$$\alpha\sqrt{k}+2\beta\sqrt{k+2k_{12}} = \gamma\sqrt{k}+2\delta\sqrt{k+2k_{12}}$$

If we impose the condition $k + 2k_{12} = c\,k$ for some $c = 0, 1, 2, \ldots$ and solve for c in terms of $\alpha, \beta, \gamma, \delta$, then we can find a degeneracy condition for this sytem.

$$\alpha\sqrt{k}+2\beta\sqrt{c\,k} = \gamma\sqrt{k}+2\delta\sqrt{c\,k}$$

$$\alpha+2\beta\sqrt{c} = \gamma+2\delta\sqrt{c}$$

$$(\alpha-\gamma)^2 = 4c\,(\delta-\beta)^2$$

$$c = \frac{1}{4}\left(\frac{\alpha-\gamma}{\delta-\beta}\right)^2,\ \delta\neq\beta$$

Since $\alpha, \beta, \gamma, \delta$ can take on any non-negative integer value, the only condition on c is that it is the square of a rational number divided by 4. Thus the spectrum is degenerate if

$$k_{12} = \frac{k}{2}\left(\frac{a^2}{4b^2}-1\right),\ \text{where } a = 0,1,2,\ldots,\ b = 1,2,3,\ldots$$

Problem 3

We can write $\mathscr{H}^1 \otimes \mathscr{H}^1 = \mathscr{H}^0 \oplus \mathscr{H}^1 \oplus \mathscr{H}^2$, and thus only need to consider the basis vectors in each of $\mathscr{H}^0$, $\mathscr{H}^1$ and $\mathscr{H}^2$. For convenience, $|a\rangle\,|b\rangle \equiv |a\rangle \otimes |b\rangle$, and kets appearing on their own are basis vectors of $\mathscr{H}^1 \otimes \mathscr{H}^1 = \mathscr{H}^0 \oplus \mathscr{H}^1 \oplus \mathscr{H}^2$, whereas those appearing as tensor products are individually basis vectors of $\mathscr{H}^1$, i.e. $|a\rangle = |a\rangle_{\mathscr{H}^1\otimes\mathscr{H}^1} = |a\rangle_{\mathscr{H}^0\oplus\mathscr{H}^1\oplus\mathscr{H}^2}$, and $|a\rangle\,|b\rangle = |a\rangle_{\mathscr{H}^1} \otimes |b\rangle_{\mathscr{H}^1}$.

$|2,2\rangle = |1,1\rangle\,|1,1\rangle$ (highest wave vector, and thus only one possible way to write in $\mathscr{H}^1 \otimes \mathscr{H}^1$)

$$J_-|2,2\rangle = \hbar\sqrt{(2+2)(2-2+1)}\,|2,1\rangle$$

$$\implies |2,1\rangle = \frac{1}{2\hbar}J_-(|1,1\rangle\,|1,1\rangle)$$

$$= \frac{1}{2\hbar}\left(\hbar\sqrt{(1+1)(1-1+1)}\,|1,0\rangle\,|1,1\rangle + \hbar\sqrt{(1+1)(1-1+1)}\,|1,1\rangle\,|1,0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\,|1,0\rangle\,|1,1\rangle + \frac{1}{\sqrt{2}}\,|1,1\rangle\,|1,0\rangle$$

$$J_-|2,1\rangle = \hbar\sqrt{(2+1)(2-1+1)}\,|2,0\rangle$$

$$\implies |2,0\rangle = \frac{1}{\hbar\sqrt{6}}J_-\left(\frac{1}{\sqrt{2}}\,|1,0\rangle\,|1,1\rangle + \frac{1}{\sqrt{2}}\,|1,1\rangle\,|1,0\rangle\right)$$

$$= \frac{1}{\hbar\sqrt{12}}\left(J_-(|1,0\rangle\,|1,1\rangle) + J_-(|1,1\rangle\,|1,0\rangle)\right)$$

$$= \frac{1}{\hbar\sqrt{12}}\Big(\hbar\sqrt{(1+0)(1-0+1)}\,|1,-1\rangle\,|1,1\rangle + \hbar\sqrt{(1+1)(1-1+1)}\,|1,0\rangle\,|1,0\rangle$$

$$+ \hbar\sqrt{(1+1)(1-1+1)}\,|1,0\rangle\,|1,0\rangle + \hbar\sqrt{(1+0)(1-0+1)}\,|1,1\rangle\,|1,-1\rangle\Big)$$

$$= \frac{1}{\sqrt{6}}\,|1,-1\rangle\,|1,1\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle\,|1,0\rangle + \frac{1}{\sqrt{6}}\,|1,1\rangle\,|1,-1\rangle$$

$$
\begin{aligned}
J_- |2,0\rangle &= \hbar\sqrt{(2+0)(2-0+1)}\, |2,-1\rangle \\
\implies |2,-1\rangle &= \frac{1}{\hbar\sqrt{6}}\, J_- \left(\frac{1}{\sqrt{6}}\, |1,-1\rangle\, |1,1\rangle + \sqrt{\frac{2}{3}}\, |1,0\rangle\, |1,0\rangle + \frac{1}{\sqrt{6}}\, |1,1\rangle\, |1,-1\rangle \right) \\
&= \frac{1}{\hbar} \left(\frac{1}{6}\, J_-(|1,-1\rangle\, |1,1\rangle) + \sqrt{\frac{2}{18}}\, J_-(|1,0\rangle\, |1,0\rangle) + \frac{1}{6}\, J_-(|1,1\rangle\, |1,-1\rangle) \right) \\
&= \frac{1}{\hbar} \left(\frac{1}{6} \left(0 + \hbar\sqrt{(1+1)(1-1+1)}\, |1,-1\rangle\, |1,0\rangle \right) \right. \\
&\quad + \frac{1}{3} \left(\hbar\sqrt{(1+0)(1-0+1)}\, |1,-1\rangle\, |1,0\rangle + \hbar\sqrt{(1+0)(1-0+1)}\, |1,0\rangle\, |1,-1\rangle \right) \\
&\quad \left. + \frac{1}{6} \left(\hbar\sqrt{(1+1)(1-1+1)}\, |1,0\rangle\, |1,-1\rangle + 0 \right) \right) \\
&= \frac{1}{\sqrt{2}}\, |1,-1\rangle\, |1,0\rangle + \frac{1}{\sqrt{2}}\, |1,0\rangle\, |1,-1\rangle
\end{aligned}
$$

$|2,-2\rangle = |1,-1\rangle\, |1,-1\rangle$ (lowest wave vector, and thus only one possible way to write in $\mathscr{H}^1 \otimes \mathscr{H}^1$)

$$
|1,1\rangle = \frac{1}{\sqrt{2}}\, |1,0\rangle\, |1,1\rangle - \frac{1}{\sqrt{2}}\, |1,1\rangle\, |1,0\rangle
$$

(since $|1,1\rangle \perp |2,1\rangle$ and $||1,1\rangle| = 1$)

$$
\begin{aligned}
J_- |1,1\rangle &= \hbar\sqrt{(1+1)(1-1+1)}\, |1,0\rangle \\
\implies |1,0\rangle &= \frac{1}{\hbar\sqrt{2}}\, J_- \left(\frac{1}{\sqrt{2}}\, |1,0\rangle\, |1,1\rangle - \frac{1}{\sqrt{2}}\, |1,1\rangle\, |1,0\rangle \right) \\
&= \frac{1}{2\hbar} \left(J_-(|1,0\rangle\, |1,1\rangle) - J_-(|1,1\rangle\, |1,0\rangle) \right) \\
&= \frac{1}{2\hbar} \left(\hbar\sqrt{(1+0)(1-0+1)}\, |1,-1\rangle\, |1,1\rangle + \hbar\sqrt{(1+1)(1-1+1)}\, |1,0\rangle\, |1,0\rangle \right. \\
&\quad \left. - \hbar\sqrt{(1+1)(1-1+1)}\, |1,0\rangle\, |1,0\rangle - \hbar\sqrt{(1+0)(1-0+1)}\, |1,1\rangle\, |1,-1\rangle \right) \\
&= \frac{1}{\sqrt{2}}\, |1,-1\rangle\, |1,1\rangle - \frac{1}{\sqrt{2}}\, |1,1\rangle\, |1,-1\rangle
\end{aligned}
$$

$$
\begin{aligned}
J_- |1,0\rangle &= \hbar\sqrt{(1+0)(1-0+1)}\, |1,-1\rangle \\
\implies |1,-1\rangle &= \frac{1}{\hbar\sqrt{2}}\, J_- \left(\frac{1}{\sqrt{2}}\, |1,-1\rangle\, |1,1\rangle - \frac{1}{\sqrt{2}}\, |1,1\rangle\, |1,-1\rangle \right) \\
&= \frac{1}{2\hbar} \left(J_-(|1,-1\rangle\, |1,1\rangle) - J_-(|1,1\rangle\, |1,-1\rangle) \right) \\
&= \frac{1}{2\hbar} \left(0 + \hbar\sqrt{(1+1)(1-1+1)}\, |1,-1\rangle\, |1,0\rangle - \hbar\sqrt{(1+1)(1-1+1)}\, |1,0\rangle\, |1,-1\rangle - 0 \right) \\
&= \frac{1}{\sqrt{2}}\, |1,-1\rangle\, |1,0\rangle + \frac{1}{\sqrt{2}}\, |1,0\rangle\, |1,-1\rangle
\end{aligned}
$$

$$
|0,0\rangle = \frac{1}{\sqrt{3}}\, |1,-1\rangle\, |1,1\rangle - \frac{1}{\sqrt{3}}\, |1,0\rangle\, |1,0\rangle + \frac{1}{\sqrt{3}}\, |1,1\rangle\, |1,-1\rangle
$$

(since $|0,0\rangle \perp |1,0\rangle$, $|0,0\rangle \perp |2,0\rangle$ and $||0,0\rangle| = 1$)

$$C^{2,2}_{1,1;1,1} = 1 \qquad C^{2,2}_{1,1;1,0} = 0 \qquad C^{2,2}_{1,1;1,-1} = 0$$
$$C^{2,2}_{1,0;1,1} = 0 \qquad C^{2,2}_{1,0;1,0} = 0 \qquad C^{2,2}_{1,0;1,-1} = 0$$
$$C^{2,2}_{1,-1;1,1} = 0 \qquad C^{2,2}_{1,-1;1,0} = 0 \qquad C^{2,2}_{1,-1;1,-1} = 0$$

$$C^{2,1}_{1,1;1,1} = 0 \qquad C^{2,1}_{1,1;1,0} = \frac{1}{\sqrt{2}} \qquad C^{2,1}_{1,1;1,-1} = 0$$
$$C^{2,1}_{1,0;1,1} = \frac{1}{\sqrt{2}} \qquad C^{2,1}_{1,0;1,0} = 0 \qquad C^{2,1}_{1,0;1,-1} = 0$$
$$C^{2,1}_{1,-1;1,1} = 0 \qquad C^{2,1}_{1,-1;1,0} = 0 \qquad C^{2,1}_{1,-1;1,-1} = 0$$

$$C^{2,0}_{1,1;1,1} = 0 \qquad C^{2,0}_{1,1;1,0} = 0 \qquad C^{2,0}_{1,1;1,-1} = \frac{1}{\sqrt{6}}$$
$$C^{2,0}_{1,0;1,1} = 0 \qquad C^{2,0}_{1,0;1,0} = \sqrt{\frac{2}{3}} \qquad C^{2,0}_{1,0;1,-1} = 0$$
$$C^{2,0}_{1,-1;1,1} = \frac{1}{\sqrt{6}} \qquad C^{2,0}_{1,-1;1,0} = 0 \qquad C^{2,0}_{1,-1;1,-1} = 0$$

$$C^{2,-1}_{1,1;1,1} = 0 \qquad C^{2,-1}_{1,1;1,0} = 0 \qquad C^{2,-1}_{1,1;1,-1} = 0$$
$$C^{2,-1}_{1,0;1,1} = 0 \qquad C^{2,-1}_{1,0;1,0} = 0 \qquad C^{2,-1}_{1,0;1,-1} = \frac{1}{\sqrt{2}}$$
$$C^{2,-1}_{1,-1;1,1} = 0 \qquad C^{2,-1}_{1,-1;1,0} = \frac{1}{\sqrt{2}} \qquad C^{2,-1}_{1,-1;1,-1} = 0$$

$$C^{2,-2}_{1,1;1,1} = 0 \qquad C^{2,-2}_{1,1;1,0} = 0 \qquad C^{2,-2}_{1,1;1,-1} = 0$$
$$C^{2,-2}_{1,0;1,1} = 0 \qquad C^{2,-2}_{1,0;1,0} = 0 \qquad C^{2,-2}_{1,0;1,-1} = 0$$
$$C^{2,-2}_{1,-1;1,1} = 0 \qquad C^{2,-2}_{1,-1;1,0} = 0 \qquad C^{2,-2}_{1,-1;1,-1} = 1$$

$$C^{1,1}_{1,1;1,1} = 0 \qquad C^{1,1}_{1,1;1,0} = -\frac{1}{\sqrt{2}} \qquad C^{1,1}_{1,1;1,-1} = 0$$
$$C^{1,1}_{1,0;1,1} = \frac{1}{\sqrt{2}} \qquad C^{1,1}_{1,0;1,0} = 0 \qquad C^{1,1}_{1,0;1,-1} = 0$$
$$C^{1,1}_{1,-1;1,1} = 0 \qquad C^{1,1}_{1,-1;1,0} = 0 \qquad C^{1,1}_{1,-1;1,-1} = 0$$

$$C^{1,0}_{1,1;1,1} = 0 \qquad C^{1,0}_{1,1;1,0} = 0 \qquad C^{1,0}_{1,1;1,-1} = -\frac{1}{\sqrt{2}}$$
$$C^{1,0}_{1,0;1,1} = 0 \qquad C^{1,0}_{1,0;1,0} = 0 \qquad C^{1,0}_{1,0;1,-1} = 0$$
$$C^{1,0}_{1,-1;1,1} = \frac{1}{\sqrt{2}} \qquad C^{1,0}_{1,-1;1,0} = 0 \qquad C^{1,0}_{1,-1;1,-1} = 0$$

$$C^{1,-1}_{1,1;1,1} = 0 \qquad C^{1,-1}_{1,1;1,0} = 0 \qquad C^{1,-1}_{1,1;1,-1} = 0$$
$$C^{1,-1}_{1,0;1,1} = 0 \qquad C^{1,-1}_{1,0;1,0} = 0 \qquad C^{1,-1}_{1,0;1,-1} = -\frac{1}{\sqrt{2}}$$
$$C^{1,-1}_{1,-1;1,1} = 0 \qquad C^{1,-1}_{1,-1;1,0} = \frac{1}{\sqrt{2}} \qquad C^{1,-1}_{1,-1;1,-1} = 0$$

$$C^{0,0}_{1,1;1,1} = 0 \qquad C^{0,0}_{1,1;1,0} = 0 \qquad C^{0,0}_{1,1;1,-1} = \frac{1}{\sqrt{3}}$$

$$C^{0,0}_{1,0;1,1} = 0 \qquad C^{0,0}_{1,0;1,0} = -\frac{1}{\sqrt{3}} \qquad C^{0,0}_{1,0;1,-1} = 0$$

$$C^{0,0}_{1,-1;1,1} = \frac{1}{\sqrt{3}} \qquad C^{0,0}_{1,-1;1,0} = 0 \qquad C^{0,0}_{1,-1;1,-1} = 0$$

Problem 4

$$\begin{aligned}
H &= J \sum_{\alpha} \left(S_1^\alpha\, S_2^\alpha + S_2^\alpha\, S_3^\alpha + S_3^\alpha\, S_1^\alpha\right) + \hbar\, B \sum_{i=1}^{3} S_i^z \\
&= J \sum_{\alpha} \frac{1}{2} \left(S_1^\alpha\, S_2^\alpha + S_2^\alpha\, S_3^\alpha + S_3^\alpha\, S_1^\alpha + S_2^\alpha\, S_1^\alpha + S_3^\alpha\, S_2^\alpha + S_1^\alpha\, S_3^\alpha \right.\\
&\quad \left. +\, S_1^\alpha\, S_1^\alpha + S_2^\alpha\, S_2^\alpha + S_3^\alpha\, S_3^\alpha - S_1^\alpha\, S_1^\alpha - S_2^\alpha\, S_2^\alpha - S_3^\alpha\, S_3^\alpha\right) + \hbar\, B\, \mathbb{S}^z \\
&\qquad\qquad (\text{since } S_i^\alpha\, S_j^\alpha = S_j^\alpha\, S_i^\alpha, \text{ and adding and subtracting } \textstyle\sum_{i=1}^{3} S_i^\alpha\, S_i^\alpha) \\
&= \frac{J}{2} \sum_{i=1}^{3} \sum_{\alpha} \left(S_i^\alpha\, S_i^\alpha\right) - \frac{J}{2} \sum_{i=1}^{3} \sum_{\alpha} S_i^\alpha\, S_i^\alpha + \hbar\, B\, \mathbb{S}^z \\
&= \frac{J}{2}\, \hat{C} - \frac{9J\, \hbar^2}{8}\, I_8 + \hbar\, B\, \mathbb{S}^z \qquad (\text{since } S_i^\alpha\, S_i^\alpha = (S^\alpha)_i^2 \text{ and } (S^\alpha)^2 = \left(\tfrac{\hbar \sigma^\alpha}{2}\right)^2 = \tfrac{\hbar^2}{4}\, I_2)
\end{aligned}$$

$$\hat{C}\, |j, m\rangle = \sum_{\alpha} \mathbb{S}^\alpha\, \mathbb{S}^\alpha\, |j, m\rangle = \hbar^2\, j\,(j+1)\, |j, m\rangle \qquad I_8\, |j, m\rangle = |j, m\rangle \qquad \mathbb{S}^z\, |j, m\rangle = \hbar\, m\, |j, m\rangle$$

$$\begin{aligned}
\implies \lambda_{j,m} &= \frac{J}{2}\, \hbar^2\, j\,(j+1) - \frac{9J\, \hbar^2}{8} + \hbar\, B\, \hbar\, m \\
&= \hbar^2 \left(J \left(\frac{j\,(j+1)}{2} - \frac{9}{8}\right) + B\, m\right)
\end{aligned}$$

$$\begin{aligned}
\mathscr{H}^{\frac{1}{2}} \otimes \mathscr{H}^{\frac{1}{2}} \otimes \mathscr{H}^{\frac{1}{2}} &= \mathscr{H}^{\frac{1}{2}} \otimes \left(\mathscr{H}^0 \oplus \mathscr{H}^1\right) \\
&= \mathscr{H}^{\frac{1}{2}} \otimes \mathscr{H}^0 \oplus \mathscr{H}^{\frac{1}{2}} \otimes \mathscr{H}^1 \\
&= \mathscr{H}^{\frac{1}{2}} \oplus \mathscr{H}^{\frac{1}{2}} \oplus \mathscr{H}^{\frac{3}{2}}
\end{aligned}$$

$$\implies j = \frac{1}{2}, \frac{3}{2},\ m = \begin{cases} -\frac{1}{2}, \frac{1}{2}, & j = \frac{1}{2} \\[4pt] -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, & j = \frac{3}{2} \end{cases}$$

Thus the spectrum is given by

$$\left\{\hbar^2 \left(J \left(\frac{j\,(j+1)}{2} - \frac{9}{8}\right) + B\, m\right)\right\} \text{ for } j = \frac{1}{2}, \frac{3}{2},\ m = \begin{cases} \pm\frac{1}{2}, & j = \frac{1}{2} \\[4pt] \pm\frac{1}{2}, \pm\frac{3}{2}, & j = \frac{3}{2} \end{cases},$$

or more explicitly,

$$\left\{\frac{\hbar^2}{4}\,(-3J - 2B)\,, \frac{\hbar^2}{4}\,(-3J + 2B)\,, \frac{\hbar^2}{4}\,(3J - 2B)\,, \frac{\hbar^2}{4}\,(3J + 2B)\,, \frac{\hbar^2}{4}\,(3J - 6B)\,, \frac{\hbar^2}{4}\,(3J + 6B)\right\}.$$