MAU34403: Quantum Mechanics I
Homework 7 due 12/11/2021

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JS Theoretical Physics

Problem 1.

(a)

$$
\begin{aligned}
|\lambda\rangle &= \sum_{n=0}^{\infty} \lambda_n \, |n\rangle \\
\lambda \, |\lambda\rangle &= a \, |\lambda\rangle \\
&= \sum_{n=0}^{\infty} \lambda_n \, a \, |n\rangle \\
&= \sum_{n=0}^{\infty} \lambda_n \sqrt{n} \, |n-1\rangle \\
&= 0 + \sum_{n=1}^{\infty} \lambda_n \sqrt{n} \, |n-1\rangle \\
&= \sum_{n=0}^{\infty} \lambda_{n+1} \sqrt{n+1} \, |n\rangle \\
\implies |\lambda\rangle &= \sum_{n=0}^{\infty} \frac{\lambda_{n+1}\sqrt{n+1}}{\lambda} \, |n\rangle \\
&= \sum_{n=0}^{\infty} \lambda_n \, |n\rangle \\
\implies \lambda_{n+1} &= \lambda_n \, \frac{\lambda}{\sqrt{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1 &= \lambda_0 \, \frac{\lambda}{\sqrt{0+1}} & \lambda_2 &= \lambda_1 \, \frac{\lambda}{\sqrt{1+1}} & \lambda_3 &= \lambda_2 \, \frac{\lambda}{\sqrt{2+1}} & \lambda_4 &= \lambda_3 \, \frac{\lambda}{\sqrt{3+1}} \\
&= \lambda_0 \, \frac{\lambda}{\sqrt{1}} & &= \lambda_0 \, \frac{\lambda^2}{\sqrt{2 \cdot 1}} & &= \lambda_0 \, \frac{\lambda^3}{\sqrt{3 \cdot 2 \cdot 1}} & &= \lambda_0 \, \frac{\lambda^4}{\sqrt{4 \cdot 3 \cdot 2 \cdot 1}}
\end{aligned}
$$

$$
\begin{aligned}
\implies \lambda_n &= \lambda_0 \, \frac{\lambda^n}{\sqrt{n!}} \\
\implies |\lambda\rangle &= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \, |n\rangle
\end{aligned}
$$

$$
\begin{aligned}
1 &= \langle\lambda|\lambda\rangle \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \langle\lambda|n\rangle \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \lambda_n^* \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \left(\lambda_0 \frac{\lambda^n}{\sqrt{n!}}\right)^* \\
&= \lambda_0 \lambda_0^* \sum_{n=0}^{\infty} \frac{(\lambda\,\lambda^*)^n}{n!} \\
&= |\lambda_0|^2 e^{|\lambda|^2} \\
\implies |\lambda_0| &= e^{-\frac{|\lambda|^2}{2}}
\end{aligned}
$$

$$
\implies |\lambda\rangle = \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \text{ where } |\lambda_0| = e^{-\frac{|\lambda|^2}{2}}
$$

(b)

$$
\begin{aligned}
|n\rangle &= \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle \\
\implies |\lambda\rangle &= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\left(\lambda\, a^\dagger\right)^n}{n!} |0\rangle \\
|\lambda\rangle &= \lambda_0\, e^{\lambda\, a^\dagger} |0\rangle \text{ where } |\lambda_0| = e^{-\frac{|\lambda|^2}{2}}
\end{aligned}
$$

(c)

$$
X = \eta\left(a^\dagger + a\right) \qquad P = \frac{i\,\hbar}{2\eta}\left(a^\dagger - a\right) \qquad \Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}
$$

$$
\begin{aligned}
\langle X\rangle &= \langle\lambda|\,\eta\left(a^\dagger + a\right)|\lambda\rangle \\
&= \eta\left(\langle\lambda|\,a^\dagger\,|\lambda\rangle + \langle\lambda|\,a\,|\lambda\rangle\right) \\
&= \eta\left(\langle\lambda|\,\lambda^*\,|\lambda\rangle + \langle\lambda|\,\lambda\,|\lambda\rangle\right) \\
&= \eta\left(\lambda^* + \lambda\right) \\
\langle X\rangle^2 &= \eta^2\left(\lambda^* + \lambda\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\langle X^2\rangle &= \langle\lambda|\,\eta^2\left(\left(a^\dagger\right)^2 + a^\dagger\,a + a\,a^\dagger + a^2\right)|\lambda\rangle \\
&= \eta^2\left(\langle\lambda|\,a^\dagger\,a^\dagger\,|\lambda\rangle + \langle\lambda|\,a^\dagger\,a\,|\lambda\rangle + \langle\lambda|\,a\,a^\dagger - a^\dagger\,a + a^\dagger\,a\,|\lambda\rangle + \langle\lambda|\,a\,a\,|\lambda\rangle\right) \\
&= \eta^2\left(\langle\lambda|\,\lambda^*\,a^\dagger\,|\lambda\rangle + \langle\lambda|\,\lambda^*\,\lambda\,|\lambda\rangle + \langle\lambda|\left[a, a^\dagger\right]|\lambda\rangle + \langle\lambda|\,a^\dagger\,a\,|\lambda\rangle + \langle\lambda|\,a\,\lambda\,|\lambda\rangle\right) \\
&= \eta^2\left(\lambda^*\,\langle\lambda|\,\lambda^*\,|\lambda\rangle + \lambda^*\,\lambda + \langle\lambda|\,I\,|\lambda\rangle + \langle\lambda|\,\lambda^*\,\lambda\,|\lambda\rangle + \lambda\,\langle\lambda|\,\lambda\,|\lambda\rangle\right) \\
&= \eta^2\left(\left(\lambda^*\right)^2 + \lambda^*\,\lambda + 1 + \lambda^*\,\lambda + \lambda^2\right) \\
&= \eta^2\left(\left(\lambda^* + \lambda\right)^2 + 1\right)
\end{aligned}
$$

$$
\implies \Delta X = \sqrt{\eta^2\left(\left(\lambda^* + \lambda\right)^2 + 1\right) - \eta^2\left(\lambda^* + \lambda\right)^2} = \eta
$$

$$
\begin{aligned}
\langle P \rangle &= \langle\lambda|\, \frac{i\,\hbar}{2\eta} \left(a^\dagger - a\right) |\lambda\rangle \\
&= \frac{i\,\hbar}{2\eta} \left(\langle\lambda|\, a^\dagger\, |\lambda\rangle - \langle\lambda|\, a\, |\lambda\rangle\right) \\
&= \frac{i\,\hbar}{2\eta} \left(\langle\lambda|\, \lambda^*\, |\lambda\rangle - \langle\lambda|\, \lambda\, |\lambda\rangle\right) \\
&= \frac{i\,\hbar}{2\eta} \left(\lambda^* - \lambda\right) \\
\langle P \rangle^2 &= -\frac{\hbar^2}{4\eta^2} \left(\lambda^* - \lambda\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\langle P^2 \rangle &= \langle\lambda| - \frac{\hbar^2}{4\eta^2} \left(\left(a^\dagger\right)^2 - a^\dagger\, a - a\, a^\dagger + a^2\right) |\lambda\rangle \\
&= -\frac{\hbar^2}{4\eta^2} \left(\langle\lambda|\, a^\dagger\, a^\dagger\, |\lambda\rangle - \langle\lambda|\, a^\dagger\, a\, |\lambda\rangle - \langle\lambda|\, a\, a^\dagger - a^\dagger\, a + a^\dagger\, a\, |\lambda\rangle + \langle\lambda|\, a\, a\, |\lambda\rangle\right) \\
&= -\frac{\hbar^2}{4\eta^2} \left(\langle\lambda|\, \lambda^*\, a^\dagger\, |\lambda\rangle - \langle\lambda|\, \lambda^*\, \lambda\, |\lambda\rangle - \langle\lambda|\, \left[a, a^\dagger\right] |\lambda\rangle - \langle\lambda|\, a^\dagger\, a\, |\lambda\rangle + \langle\lambda|\, a\, \lambda\, |\lambda\rangle\right) \\
&= -\frac{\hbar^2}{4\eta^2} \left(\lambda^*\, \langle\lambda|\, \lambda^*\, |\lambda\rangle - \lambda^*\, \lambda - \langle\lambda|\, I\, |\lambda\rangle - \langle\lambda|\, \lambda^*\, \lambda\, |\lambda\rangle + \lambda\, \langle\lambda|\, \lambda\, |\lambda\rangle\right) \\
&= -\frac{\hbar^2}{4\eta^2} \left(\left(\lambda^*\right)^2 - \lambda^*\, \lambda - 1 - \lambda^*\, \lambda + \lambda^2\right) \\
&= -\frac{\hbar^2}{4\eta^2} \left(\left(\lambda^* - \lambda\right)^2 - 1\right)
\end{aligned}
$$

$$
\implies \Delta P = \sqrt{-\frac{\hbar^2}{4\eta^2} \left(\left(\lambda^* - \lambda\right)^2 - 1\right) + \frac{\hbar^2}{4\eta^2} \left(\lambda^* - \lambda\right)^2} = \frac{\hbar}{2\eta}
$$

$$
\Delta X\, \Delta P = \eta\, \frac{\hbar}{2\eta} = \frac{\hbar}{2} \implies \text{uncertainty relation verified}
$$

(d)

$$
\begin{aligned}
\psi_\lambda(x) &= \langle x|\lambda\rangle \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \langle x|n\rangle \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \psi_n(x) \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}\, \frac{1}{\sqrt{2^n\, n!}}\, H_n\!\left(\frac{x}{\eta\sqrt{2}}\right) \psi_0(x) \\
&= \lambda_0 \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{\sqrt{2}}\right)^n}{n!}\, H_n\!\left(\frac{x}{\eta\sqrt{2}}\right) \frac{1}{\sqrt{\eta\sqrt{2\pi}}} \exp\!\left(-\frac{x^2}{4\eta^2}\right) \\
&= \frac{\lambda_0}{\sqrt{\eta\sqrt{2\pi}}} \exp\!\left(-\frac{x^2}{4\eta^2}\right) \sum_{n=0}^{\infty} H_n\!\left(\frac{x}{\eta\sqrt{2}}\right) \frac{\left(\frac{\lambda}{\sqrt{2}}\right)^n}{n!} \\
&= \frac{\lambda_0}{\sqrt{\eta\sqrt{2\pi}}} \exp\!\left(-\frac{x^2}{4\eta^2}\right) \exp\!\left(2\, \frac{x}{\eta\sqrt{2}}\, \frac{\lambda}{\sqrt{2}} - \left(\frac{\lambda}{\sqrt{2}}\right)^2\right) \\
\psi_\lambda(x) &= \frac{\lambda_0}{\sqrt{\eta\sqrt{2\pi}}} \exp\!\left(-\frac{x^2}{4\eta^2} + \frac{x\,\lambda}{\eta} - \frac{\lambda^2}{2}\right) \quad \text{where } |\lambda_0| = e^{-\frac{|\lambda|^2}{2}}
\end{aligned}
$$

$$
\text{Sum}\left[\text{HermiteH}[n, a]\, \frac{b^n}{n!}, \{n, 0, \infty\}\right]
$$

$$
e^{2\,a\,b - b^2}
$$

(e)

$$
\begin{aligned}
|\lambda\rangle &= \lambda_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \, |n\rangle \\
&= \sum_{n=0}^{\infty} f(n) \, |n\rangle \\
\implies f(n) &= \frac{\lambda_0 \, \lambda^n}{\sqrt{n!}} \\
|f(n)|^2 &= \frac{|\lambda_0|^2 \, |\lambda^n|^2}{\left|\sqrt{n!}\right|^2} \\
&= \frac{\left|\lambda^2\right|^n}{n!} \, e^{-|\lambda|^2}
\end{aligned}
$$

A Poisson distribution is of the form

$$
P(n) = \frac{\bar{n}^n}{n!} \, e^{-\bar{n}}.
$$

$|f(n)|^2$ is a Poisson distribution if it takes this form, i.e. if $\left|\lambda^2\right| = |\lambda|^2$, which is true, and so $|f(n)|^2$ is a Poisson distribution.

From inspection of the Poisson distribution, we can see that $\bar{n} = |\lambda|^2$. The expectation value of $a^\dagger \, a$ can be found as

$$
\langle\lambda| \, a^\dagger \, a \, |\lambda\rangle = \langle\lambda| \, \lambda^* \, \lambda \, |\lambda\rangle = |\lambda|^2,
$$

and so $\bar{n} = \langle a^\dagger \, a\rangle = |\lambda|^2$.

The most probable value of n can be found by finding the peak of the Poisson distribution. This occurs at n_{mp} when $P(n_{\text{mp}}) \geqslant P(n_{\text{mp}} + 1)$, and thus n_{mp} can be found as

$$
\begin{aligned}
|f(n_{\text{mp}} + 1)|^2 &\geqslant |f(n_{\text{mp}})|^2 \\
\frac{\left|\lambda^2\right|^{n_{\text{mp}}+1}}{(n_{\text{mp}} + 1)!} \, e^{-|\lambda|^2} &\geqslant \frac{\left|\lambda^2\right|^{n_{\text{mp}}}}{n_{\text{mp}}!} \, e^{-|\lambda|^2} \\
\frac{|\lambda|^2}{n_{\text{mp}} + 1} &\geqslant 1 \\
n_{\text{mp}} &\geqslant |\lambda|^2 - 1 \\
n_{\text{mp}} &= \left\lfloor |\lambda|^2 \right\rfloor
\end{aligned}
$$

Problem 2.

(a)

$$
\begin{aligned}
V(x) \, \psi(x) &= \frac{\hbar^2}{2m} \, \psi''(x) + E \, \psi(x) \\
\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} V(x) \, \psi(x) \, dx &= \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \left(\frac{\hbar^2}{2m} \, \psi''(x) + E \, \psi(x) \right) dx \\
\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \nu \, \delta(x) \, \psi(x) \, dx &= \frac{\hbar^2}{2m} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \psi''(x) \, dx + E \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \psi(x) \, dx \\
\lim_{\varepsilon \to 0} \nu \, \psi(0) &= \frac{\hbar^2}{2m} \lim_{\varepsilon \to 0} \psi'(x) \bigg|_{-\varepsilon}^{\varepsilon} + 0 \qquad \text{(since } \psi(x) \text{ is continuous at } x = 0\text{)} \\
\psi(0) &= \frac{\hbar^2}{2m \, \nu} \left(\psi'_R(0) - \psi'_L(0) \right) \\
&\qquad \text{(where } \psi_L(x) \text{ and } \psi_R(x) \text{ are the wavefunctions either side of } x = 0\text{)}
\end{aligned}
$$

Assume $\psi(x)$ takes the form

$$\psi(x) = \begin{cases} A_L \sin(k\,x + \phi_L)\,, & -a \leqslant x \leqslant 0 \\ A_R \sin(k\,x + \phi_R)\,, & 0 \leqslant x \leqslant a \\ 0, & |x| \geqslant a \end{cases}.$$

$$\begin{aligned} 0 &= \psi_L(-a) \\ &= A_L \sin(-k\,a + \phi_L) \\ \implies \phi_L &= k\,a \end{aligned} \qquad\qquad \begin{aligned} 0 &= \psi_R(a) \\ &= A_R \sin(k\,a + \phi_R) \\ \implies \phi_R &= -k\,a \end{aligned}$$

$$\implies \psi(x) = \begin{cases} A_L \sin(k\,x + k\,a)\,, & -a \leqslant x \leqslant 0 \\ A_R \sin(k\,x - k\,a)\,, & 0 \leqslant x \leqslant a \\ 0, & |x| \geqslant a \end{cases}$$

If $\psi(x)$ is odd, then we must have $\psi(-x) = -\psi(x)$ for all x. Thus, $\psi(0) = \psi(-0) = -\psi(0)$ and so we have $\psi(0) = 0$. We therefore have $A_L \sin(k\,a) = A_R \sin(-k\,a) = 0$, and therefore $k\,a = n\,\pi$, where $n = 0, 1, 2, \ldots$. Since $n = 0$ corresponds to an energy of 0 and thus a non-normalisable constant 0 wavefunction, we only consider positive non-zero integers for n. We then have

$$k\,a = n\,\pi \implies E = \frac{n^2\,\pi^2\,\hbar^2}{2m\,a^2},\; n = 1, 2, \ldots$$

for odd wavefunctions $\psi(x)$.

Now assume $\psi(x)$ is even. Assuming $\psi(0) \neq 0$ (as this would result in the same energy quantisation condition above), we have that

$$A_L \sin(k\,a) = A_R \sin(-k\,a) = -A_R \sin(k\,a) \implies A \equiv A_L = -A_R.$$

Our wavefunction thus takes the form

$$\psi(x) = \begin{cases} A \sin(k\,x + k\,a)\,, & -a \leqslant x \leqslant 0 \\ -A \sin(k\,x - k\,a)\,, & 0 \leqslant x \leqslant a \\ 0, & |x| \geqslant a \end{cases}.$$

From the expression relating $\psi(0)$, $\psi_R'(0)$ and $\psi_L'(0)$ we have

$$\begin{aligned} \psi(0) &= \frac{\hbar^2}{2m\,\nu}\,(\psi_R'(0) - \psi_L'(0)) \\ A \sin(k\,a) &= \frac{\hbar^2}{2m\,\nu}\,(-k\,A\cos(-k\,a) - k\,A\cos(k\,a)) \\ \sin(k\,a) &= -\frac{k\,\hbar^2}{m\,\nu}\,\cos(k\,a) \\ \tan(k\,a) &= -\frac{k\,\hbar^2}{m\,\nu} \end{aligned}$$

Our quantisation conditions are therefore

$$\psi(x) \text{ odd} \implies E = \frac{n^2\,\pi^2\,\hbar^2}{2m\,a^2},\; n = 1, 2, \ldots \qquad\qquad \psi(x) \text{ even} \implies \tan(k\,a) = -\frac{k\,\hbar^2}{m\,\nu}$$

(b)

(i)

For small ν we have that for even $\psi(x)$, $-\tan(k\,a)$ is very large, and thus tends to infinity. Thus $k\,a = \frac{n\,\pi}{2}$ for $n = 1, 3, 5, \ldots$, which leads to $E = \frac{\left(\frac{n}{2}\right)^2 \pi^2\,\hbar^2}{2m\,a^2}$ for $n = 1, 3, 5, \ldots$ or $E = \frac{\left(n + \frac{1}{2}\right)^2 \pi^2\,\hbar^2}{2m\,a^2}$ for $n = 0, 1, 2, \ldots$. Combining with the energy values for odd $\psi(x)$ gives us $E = \frac{n^2\,\pi^2\,\hbar^2}{2m\,a^2}$ for $n = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, or $E = \frac{n^2\,\pi^2\,\hbar^2}{8m\,a^2}$ for $n = 1, 2, \ldots$. Since this energy does not have any ν dependence, we can immediately say that

$$E_n^{(0)} = \frac{n^2\,\pi^2\,\hbar^2}{8m\,a^2}, \qquad\qquad E_n^{(1)} = 0.$$

Thus for small enough ν, the energy levels match those of the infinite square well with no barrier.

(ii)

For large ν we have that for even $\psi(x)$, $\tan(k\,a)$ is very small, i.e. approximately 0. Thus $k\,a = n\,\pi$ for $n = 0, 1, 2, \ldots$. This leads to the same energy as that for odd $\psi(x)$, and thus $E = \frac{n^2\,\pi^2\,\hbar^2}{2m\,a^2}$ for $n = 1, 2, \ldots$. Similarly, since these energy values have no ν dependence,

$$E_n^{(\infty)} = \frac{n^2\,\pi^2\,\hbar^2}{2m\,a^2}, \qquad\qquad E_n^{(-1)} = 0.$$

Thus for large enough ν, only the odd wavefunctions are possible solutions for the Schrödinger equation, since the odd functions are not affected by the barrier (as $\psi(0) = 0$) yet the even functions cannot pass through.

(c)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi(x)|^2\,dx \\
&= \int_{-a}^{0} |A|^2 \sin^2(k\,x + k\,a)\,dx + \int_{0}^{a} |A|^2 \sin^2(k\,x - k\,a)\,dx \\
&= |A|^2\,\frac{2k\,a - \sin(2k\,a)}{2k} \\
\implies |A| &= \frac{1}{\sqrt{a - \frac{\sin(2k\,a)}{2k}}} \\
&= \frac{1}{\sqrt{a - \frac{1}{2k}\,\frac{2\tan(k\,a)}{1+\tan^2(k\,a)}}} \\
&= \frac{1}{\sqrt{a + \frac{1}{k}\,\frac{k\,\hbar^2}{m\,\nu}\,\frac{1}{1+\frac{k^2\,\hbar^4}{m^2\,\nu^2}}}} \\
&= \frac{1}{\sqrt{a + \frac{\hbar^2}{m\,\nu + \frac{k^2\,\hbar^4}{m\,\nu}}}} \\
&= \frac{1}{\sqrt{a + \frac{\hbar^2\,m\,\nu}{m^2\,\nu^2 + k^2\,\hbar^4}}}
\end{aligned}$$

```
Integrate[(Sin[k x + k a])^2, {x, -a, 0}] +
  Integrate[(Sin[k x - k a])^2, {x, 0, a}] // FullSimplify
```

$$a - \frac{\mathrm{Sin}[2\,a\,k]}{2\,k}$$

$$\implies \psi(x) = \begin{cases} A\sin(k\,x + k\,a)\,, & -a \leqslant x \leqslant 0 \\ -A\sin(k\,x - k\,a)\,, & 0 \leqslant x \leqslant a \\ 0, & |x| \geqslant a \end{cases}, \text{ where } |A| = \frac{1}{\sqrt{a + \frac{\hbar^2\,m\,\nu}{m^2\,\nu^2 + k^2\,\hbar^4}}}$$

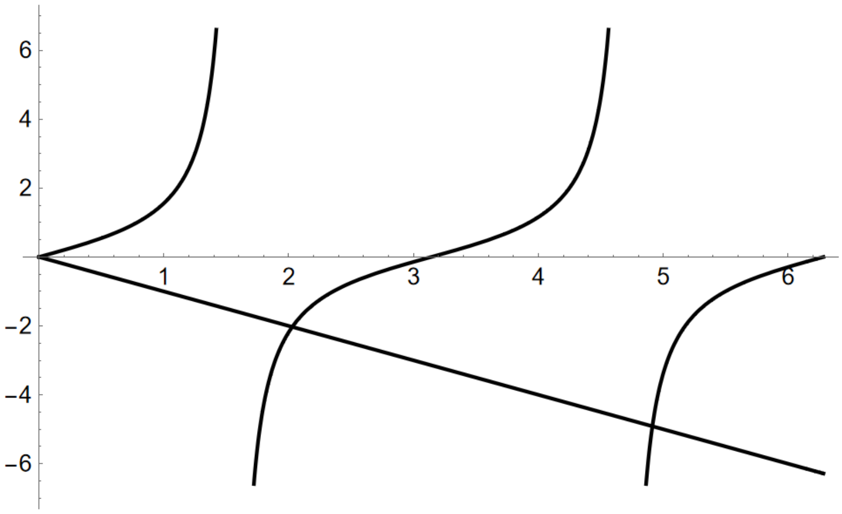
$$a = 1,\ m = 1,\ \hbar = 1,\ \nu = 1 \implies \psi(x) = \begin{cases} A\sin(k\,(x+1))\,, & -1 \leqslant x \leqslant 0 \\ -A\sin(k\,(x-1))\,, & 0 \leqslant x \leqslant 1 \\ 0, & |x| \geqslant 1 \end{cases}, \text{ where } |A| = \frac{1}{\sqrt{1 + \frac{1}{1+k^2}}}$$

$$\text{and } \tan(k) = -k \implies k \approx 2.02876 \implies E = \frac{k^2}{2} \approx 2.05793$$

$$a=1,\ m=1,\ \hbar=1,\ \nu=10 \implies \psi(x)=\begin{cases} A\sin(k\,(x+1))\,, & -1\leqslant x\leqslant 0\\ -A\sin(k\,(x-1))\,, & 0\leqslant x\leqslant 1\\ 0, & |x|\geqslant 1\end{cases}\ ,\ \text{where } |A|=\frac{1}{\sqrt{1+\frac{10}{100+k^2}}}$$

$$\text{and } \tan(k)=-\frac{k}{10} \implies k\approx 2.86277 \implies E=\frac{k^2}{2}\approx 4.09773$$

```
tank = Plot[Tan[k], {k, 0., 2 π}, PlotStyle → Black];
minusk1 = Plot[-k, {k, 0., 2 π}, PlotStyle → Black];
Show[tank, minusk1]
```



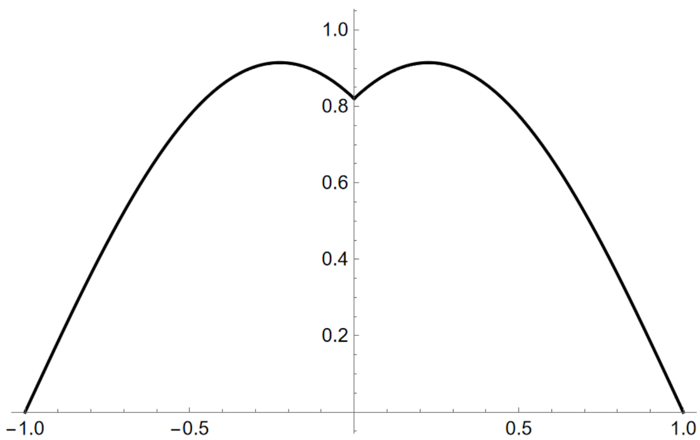

```
FindRoot[Tan[x] == -x, {x, 2}]
```

{x → 2.02876}

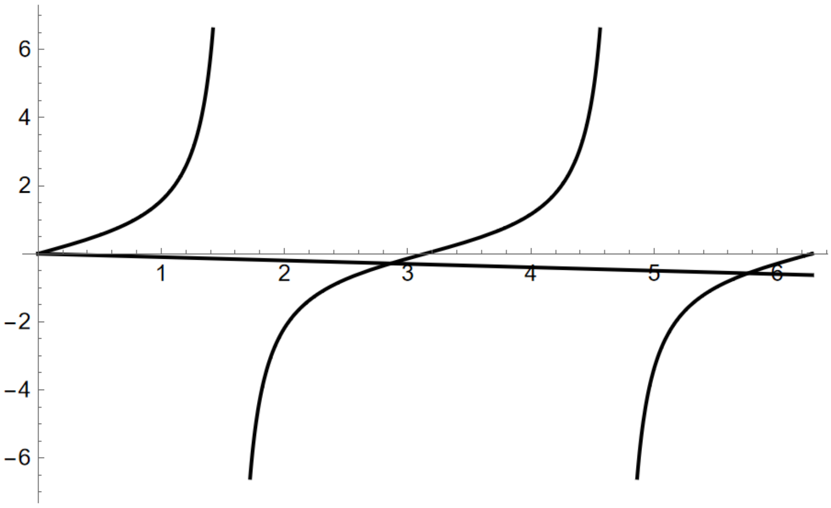
```
k = 2.028757838110434;
```

$$A = \frac{1}{\sqrt{1+\frac{1}{1+k^2}}};$$

```
leftpsi1 = Plot[A Sin[k (x + 1)], {x, -1, 0}, PlotStyle → Black];
rightpsi1 = Plot[-A Sin[k (x - 1)], {x, 0, 1}, PlotStyle → Black];
Show[leftpsi1, rightpsi1, PlotRange → {{-1, 1}, {0, 1}}]
```



```
tank = Plot[Tan[k], {k, 0., 2 π}, PlotStyle → Black];
minusk10 = Plot[-k/10, {k, 0., 2 π}, PlotStyle → Black];
Show[tank, minusk10]
```




```
FindRoot[Tan[x] == -x/10, {x, 3}]
```

{x → 2.86277}

```
k = 2.8627725875152072;
```

$$A = \frac{1}{\sqrt{1+\frac{10}{100+k^2}}};$$

```
leftpsi10 = Plot[A Sin[k (x + 1)], {x, -1, 0}, PlotStyle → Black];
rightpsi10 = Plot[-A Sin[k (x - 1)], {x, 0, 1}, PlotStyle → Black];
Show[leftpsi10, rightpsi10, PlotRange → {{-1, 1}, {0, 1}}]
```