MAU34403: Quantum Mechanics I
Homework 6 due 05/11/2021

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JS Theoretical Physics

Problem 1

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{\partial(\sqrt{\rho})}{\partial t}\exp\left(\frac{i\,S}{\hbar}\right) + \sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right)\frac{i}{\hbar}\,\frac{\partial S}{\partial t} \\
&= \frac{1}{\sqrt{\rho}}\frac{\partial(\sqrt{\rho})}{\partial t}\sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right) + \frac{i}{\hbar}\,\frac{\partial S}{\partial t}\sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right) \\
&= \frac{\partial\log\sqrt{\rho}}{\partial t}\,\psi + \frac{i}{\hbar}\,\frac{\partial S}{\partial t}\,\psi
\end{aligned}$$

$$\begin{aligned}
\nabla^2\psi &= \vec{\nabla}\cdot\vec{\nabla}(\psi) \\
&= \vec{\nabla}\cdot\left(\vec{\nabla}(\sqrt{\rho})\exp\left(\frac{i\,S}{\hbar}\right) + \sqrt{\rho}\,\vec{\nabla}\left(\exp\left(\frac{i\,S}{\hbar}\right)\right)\right) \\
&= \nabla^2(\sqrt{\rho})\exp\left(\frac{i\,S}{\hbar}\right) + \vec{\nabla}(\sqrt{\rho})\,\vec{\nabla}\left(\exp\left(\frac{i\,S}{\hbar}\right)\right) + \vec{\nabla}(\sqrt{\rho})\,\vec{\nabla}\left(\exp\left(\frac{i\,S}{\hbar}\right)\right) + \sqrt{\rho}\,\nabla^2\left(\exp\left(\frac{i\,S}{\hbar}\right)\right) \\
&= \frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right) + \left(2\vec{\nabla}(\sqrt{\rho})\right)\left(\frac{i}{\hbar}\,\vec{\nabla}(S)\,\exp\left(\frac{i\,S}{\hbar}\right)\right) + \sqrt{\rho}\,\vec{\nabla}\cdot\left(\frac{i}{\hbar}\,\vec{\nabla}(S)\,\exp\left(\frac{i\,S}{\hbar}\right)\right) \\
&= \frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi + \frac{2i}{\hbar}\,\frac{\vec{\nabla}(\sqrt{\rho})}{\sqrt{\rho}}\,\vec{\nabla}(S)\,\sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right) + \sqrt{\rho}\left(\frac{i}{\hbar}\nabla^2(S)\,\exp\left(\frac{i\,S}{\hbar}\right) + \frac{i}{\hbar}\,\vec{\nabla}(S)\,\vec{\nabla}\left(\exp\left(\frac{i\,S}{\hbar}\right)\right)\right) \\
&= \frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi + \frac{2i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\,\psi + \frac{i}{\hbar}\nabla^2(S)\sqrt{\rho}\,\exp\left(\frac{i\,S}{\hbar}\right) + \sqrt{\rho}\,\frac{i}{\hbar}\,\vec{\nabla}(S)\frac{i}{\hbar}\,\vec{\nabla}(S)\exp\left(\frac{i\,S}{\hbar}\right) \\
&= \frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi + \frac{2i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\,\psi + \frac{i}{\hbar}\nabla^2(S)\psi - \frac{1}{\hbar^2}\vec{\nabla}(S)\,\vec{\nabla}(S)\psi
\end{aligned}$$

$$\begin{aligned}
i\,\hbar\,\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2(\psi) + V\,\psi \\
i\,\hbar\left(\frac{\partial\log\sqrt{\rho}}{\partial t}\,\psi + \frac{i}{\hbar}\,\frac{\partial S}{\partial t}\,\psi\right) &= -\frac{\hbar^2}{2m}\left(\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi + \frac{2i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\,\psi + \frac{i}{\hbar}\nabla^2(S)\psi - \frac{1}{\hbar^2}\vec{\nabla}(S)\,\vec{\nabla}(S)\psi\right) \\
\left(-\frac{\partial S}{\partial t} + i\left(\hbar\,\frac{\partial\log\sqrt{\rho}}{\partial t}\right)\right)\psi &= \left(-\frac{\hbar^2}{2m}\,\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} + \frac{1}{2m}\,\vec{\nabla}(S)\,\vec{\nabla}(S) + V + i\left(-\frac{\hbar}{m}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S) - \frac{\hbar}{2m}\,\nabla^2(S)\right)\right)\psi
\end{aligned}$$

$$\begin{aligned}
\hbar\,\frac{\partial\log\sqrt{\rho}}{\partial t} &= -\frac{\hbar}{m}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S) - \frac{\hbar}{2m}\,\nabla^2(S) & -\frac{\partial S}{\partial t} &= -\frac{\hbar^2}{2m}\,\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} + \frac{1}{2m}\,\vec{\nabla}(S)\,\vec{\nabla}(S) \\
\implies \frac{\partial\log\sqrt{\rho}}{\partial t} &= -\frac{1}{2m}\left(\nabla^2(S) + 2\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\right) & 0 &= \frac{\partial S}{\partial t} + \frac{1}{2m}\,\vec{\nabla}(S)\,\vec{\nabla}(S) + V - \frac{\hbar^2}{2m}\,\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}},
\end{aligned}$$

as required.

$$
\begin{aligned}
-\vec{\nabla}\cdot\vec{J} &= -\vec{\nabla}\cdot\frac{i\hbar}{2m}\left(\vec{\nabla}(\psi^*)\,\psi-\psi^*\vec{\nabla}(\psi)\right)\\
&= -\frac{i\hbar}{2m}\left(\nabla^2(\psi^*)\,\psi+\vec{\nabla}(\psi^*)\,\vec{\nabla}(\psi)-\vec{\nabla}(\psi^*)\,\vec{\nabla}(\psi)-\psi^*\nabla^2(\psi)\right)\\
&= -\frac{i\hbar}{2m}\left(\nabla^2(\psi^*)\,\psi-\psi^*\nabla^2(\psi)\right)\\
&= -\frac{i\hbar}{2m}\left(\left(\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi-\frac{2i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\,\psi-\frac{i}{\hbar}\nabla^2(S)\psi-\frac{1}{\hbar^2}\vec{\nabla}(S)\,\vec{\nabla}(S)\psi\right)\psi^*\,\psi\right.\\
&\quad\left.-\,\psi^*\left(\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}}\,\psi+\frac{2i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\,\psi+\frac{i}{\hbar}\nabla^2(S)\psi-\frac{1}{\hbar^2}\vec{\nabla}(S)\,\vec{\nabla}(S)\psi\right)\psi\right)\\
&= -\frac{i\hbar}{2m}\left(-\frac{4i}{\hbar}\,\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)-\frac{2i}{\hbar}\,\nabla^2(S)\right)\sqrt{\rho}\,\exp\!\left(-\frac{i\,S}{\hbar}\right)\sqrt{\rho}\,\exp\!\left(\frac{i\,S}{\hbar}\right)\\
&= -\frac{\rho}{m}\left(2\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)+\nabla^2(S)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\log\sqrt{\rho}}{\partial t} &= \frac{\partial\rho}{\partial t}\,\frac{\partial\log\sqrt{\rho}}{\partial\rho}=\frac{\partial\rho}{\partial t}\,\frac{1}{\sqrt{\rho}}\,\frac{1}{2\sqrt{\rho}}=\frac{\partial\rho}{\partial t}\,\frac{1}{2\rho}\\
\implies \frac{\partial\rho}{\partial t} &= 2\rho\left(-\frac{1}{2m}\left(\nabla^2(S)+2\vec{\nabla}(\log\sqrt{\rho})\,\vec{\nabla}(S)\right)\right)\\
&= -\vec{\nabla}\cdot\vec{J},
\end{aligned}
$$

and so the first equation is analogous to the continuity equation.

Problem 2

$$
\begin{aligned}
\hat{F}(\alpha) &\equiv -e^{\alpha\hat{A}}\,\frac{d}{dt}e^{-\alpha\hat{A}}\\
-e^{\hat{A}}\,\frac{d}{dt}e^{-\hat{A}} &= \hat{F}(1)\\
&= \sum_{n=0}^{\infty}\frac{\hat{F}^{(n)}(\beta)}{n!}\,(\alpha-\beta)^n\Bigg|_{\alpha=1}\\
&= \sum_{n=0}^{\infty}\frac{\hat{F}^{(n)}(0)}{n!} \qquad \text{(expanding about } \beta=0)
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}(\alpha) &= -e^{\alpha\hat{A}}\,\frac{d}{dt}e^{-\alpha\hat{A}}\\
\hat{F}(0) &= -1\,\frac{d}{dt}(1)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}'(\alpha) &= \frac{d}{d\alpha}\left(-e^{\alpha\hat{A}}\,\frac{d}{dt}e^{-\alpha\hat{A}}\right)\\
&= -e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}e^{-\alpha\hat{A}}+e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}}\\
\hat{F}'(0) &= -\hat{A}\,\frac{d}{dt}(1)+\frac{d}{dt}\hat{A}\\
&= \frac{d\hat{A}}{dt}
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}''(\alpha) &= \frac{d}{d\alpha}\left(-e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}}\right) \\
&= -e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\hat{A}\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}} \\
&= -e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}e^{-\alpha\hat{A}} + 2e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}} \\
\hat{F}''(0) &= -\hat{A}^2\,\frac{d}{dt}(1) + 2\hat{A}\,\frac{d}{dt}\hat{A} - \frac{d}{dt}\hat{A}^2 \\
&= 2\hat{A}\,\frac{d\hat{A}}{dt} - \frac{d\hat{A}}{dt}\,\hat{A} - \hat{A}\,\frac{d\hat{A}}{dt} \\
&= \hat{A}\,\frac{d\hat{A}}{dt} - \frac{d\hat{A}}{dt}\,\hat{A} \\
&= \left[\hat{A}, \frac{d\hat{A}}{dt}\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}'''(\alpha) &= \frac{d}{d\alpha}\left(-e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}e^{-\alpha\hat{A}} + 2e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}}\right) \\
&= -e^{\alpha\hat{A}}\,\hat{A}^3\,\frac{d}{dt}e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} + 2e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} - 2e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}^3\,e^{\alpha\hat{A}} \\
&= -e^{\alpha\hat{A}}\,\hat{A}^3\,\frac{d}{dt}e^{-\alpha\hat{A}} + 3e^{\alpha\hat{A}}\,\hat{A}^2\,\frac{d}{dt}\hat{A}\,e^{-\alpha\hat{A}} - 3e^{\alpha\hat{A}}\,\hat{A}\,\frac{d}{dt}\hat{A}^2\,e^{-\alpha\hat{A}} + e^{\alpha\hat{A}}\,\frac{d}{dt}\hat{A}^3\,e^{\alpha\hat{A}} \\
\hat{F}'''(0) &= -\hat{A}^3\,\frac{d}{dt}(1) + 3\hat{A}^2\,\frac{d}{dt}\hat{A} - 3\hat{A}\,\frac{d}{dt}\hat{A}^2 + \frac{d}{dt}\hat{A}^3 \\
&= 0 + 3\hat{A}^2\,\frac{d\hat{A}}{dt} - 3\hat{A}\left(\frac{d\hat{A}}{dt}\,\hat{A} + \hat{A}\,\frac{d\hat{A}}{dt}\right) + \frac{d\left(\hat{A}^2\right)}{dt}\,\hat{A} + \hat{A}^2\,\frac{d\hat{A}}{dt} \\
&= -3\hat{A}\,\frac{d\hat{A}}{dt}\,\hat{A} + \frac{d\hat{A}}{dt}\,\hat{A}^2 + \hat{A}\,\frac{d\hat{A}}{dt}\,\hat{A} + \hat{A}^2\,\frac{d\hat{A}}{dt} \\
&= \hat{A}^2\,\frac{d\hat{A}}{dt} - 2\hat{A}\,\frac{d\hat{A}}{dt}\,\hat{A} + \frac{d\hat{A}}{dt}\,\hat{A}^2 \\
&= \hat{A}\left(\hat{A}\,\frac{d\hat{A}}{dt} - \frac{d\hat{A}}{dt}\,\hat{A}\right) - \left(\hat{A}\,\frac{d\hat{A}}{dt} - \frac{d\hat{A}}{dt}\,\hat{A}\right)\hat{A} \\
&= \hat{A}\left[\hat{A}, \frac{d\hat{A}}{dt}\right] - \left[\hat{A}, \frac{d\hat{A}}{dt}\right]\hat{A} \\
&= \left[\hat{A}, \left[\hat{A}, \frac{d\hat{A}}{dt}\right]\right]
\end{aligned}
$$

$$
\implies \hat{F}^{(n)}(0) = \begin{cases} 0, & n = 0 \\ \dfrac{d\hat{A}}{dt}, & n = 1 \\ \underbrace{\left[\hat{A}, \left[\hat{A}, \cdots \left[\hat{A}\right.\right.\right.}_{n-1}, \left.\left.\left.\frac{d\hat{A}}{dt}\right]\right]\right], & n \geqslant 2 \end{cases}
$$

$$
\begin{aligned}
-e^{\hat{A}}\,\frac{d}{dt}e^{-\hat{A}} &= \sum_{n=0}^{\infty}\frac{\hat{F}^{(n)}(0)}{n!} \\
&= 0 + \frac{d\hat{A}}{dt} + \sum_{n=2}^{\infty}\frac{1}{n!}\underbrace{\left[\hat{A},\left[\hat{A},\cdots\left[\hat{A},\right.\right.\right.}_{n-1}\left.\left.\left.\frac{d\hat{A}}{dt}\right]\right]\right] \\
&= \frac{d\hat{A}}{dt} + \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\underbrace{\left[\hat{A},\left[\hat{A},\cdots\left[\hat{A},\right.\right.\right.}_{n}\left.\left.\left.\frac{d\hat{A}}{dt}\right]\right]\right]
\end{aligned}
$$

Problem 3

1.

$$
\begin{aligned}
\left[L^{\vec{m}}, L^{\vec{n}}\right] &= \left[\sum_{\alpha} m^{\alpha}\,L^{\alpha}, \sum_{\beta} n^{\beta}\,L^{\beta}\right] \\
&= \sum_{\alpha,\beta} m^{\alpha}\,n^{\beta}\left[L^{\alpha}, L^{\beta}\right] \\
&= \sum_{\alpha,\beta} m^{\alpha}\,n^{\beta}\sum_{\gamma} i\,\hbar\,\epsilon^{\alpha\beta\gamma}\,L^{\gamma} \\
&= i\,\hbar\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\,m^{\alpha}\,n^{\beta}\,L^{\gamma} \\
&= i\,\hbar\,(\vec{m}\times\vec{n})\cdot\vec{L} \\
&= i\,\hbar\,\vec{L}^{\vec{m}\times\vec{n}}
\end{aligned}
$$

$$
\begin{aligned}
\left[L^{\vec{n}}, \vec{X}\right] &= \left[\sum_{\alpha} n^{\alpha}\,L^{\alpha}, \sum_{\beta} X^{\beta}\,\hat{e}_{\beta}\right] \\
&= \sum_{\alpha,\beta} n^{\alpha}\,\hat{e}_{\beta}\left[L^{\alpha}, X^{\beta}\right] \\
&= \sum_{\alpha,\beta} n^{\alpha}\,\hat{e}_{\beta}\sum_{\gamma} i\,\hbar\,\epsilon^{\alpha\beta\gamma}\,X^{\gamma} \\
&= i\,\hbar\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\,n^{\alpha}\,\hat{e}_{\beta}\,X^{\gamma} \\
&= i\,\hbar\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\,X^{\alpha}\,n^{\beta}\,\hat{e}_{\gamma} \\
&= i\,\hbar\,\vec{X}\times\vec{n}
\end{aligned}
$$

$$
\begin{aligned}
\left[L^{\vec{n}}, \vec{P}\right] &= \left[\sum_{\alpha} n^{\alpha}\,L^{\alpha}, \sum_{\beta} P^{\beta}\,\hat{e}_{\beta}\right] \\
&= \sum_{\alpha,\beta} n^{\alpha}\,\hat{e}_{\beta}\left[L^{\alpha}, P^{\beta}\right] \\
&= \sum_{\alpha,\beta} n^{\alpha}\,\hat{e}_{\beta}\sum_{\gamma} i\,\hbar\,\epsilon^{\alpha\beta\gamma}\,P^{\gamma} \\
&= i\,\hbar\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\,n^{\alpha}\,\hat{e}_{\beta}\,P^{\gamma} \\
&= i\,\hbar\sum_{\alpha,\beta,\gamma}\epsilon^{\alpha\beta\gamma}\,P^{\alpha}\,n^{\beta}\,\hat{e}_{\gamma} \\
&= i\,\hbar\,\vec{P}\times\vec{n}
\end{aligned}
$$

2.

$$
\begin{aligned}
\vec{X}\left(\vec{\vartheta}\right) &= R^{\dagger}\left(\vec{\vartheta}\right)\vec{X}\,R\left(\vec{\vartheta}\right) \\
&= \exp\left(\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}}\right)\vec{X}\exp\left(-\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}}\right) \\
&= \vec{X} + \sum_{m=1}^{\infty}\frac{1}{m!}\underbrace{\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\cdots\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\right.\right.\right.}_{m}\left.\left.\left.\vec{X}\right]\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}}, \vec{X}\right] &= \frac{i\,\vartheta}{\hbar}\left[L^{\vec{n}}, \vec{X}\right] \\
&= -\vartheta\,\vec{X}\times\vec{n}
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\vec X\right]\right] &= -\frac{i\,\vartheta^2}{\hbar}\left[L^{\vec n},\vec X\times\vec n\right]\\
&= -\frac{i\,\vartheta^2}{\hbar}\left[\sum_\alpha n^\alpha\,L^\alpha,\ \sum_{\beta,\gamma,\lambda}\epsilon^{\beta\gamma\delta}\,X^\beta\,n^\gamma\,\hat e_\lambda\right]\\
&= -\frac{i\,\vartheta^2}{\hbar}\sum_{\alpha,\beta,\gamma,\lambda}\epsilon^{\beta\gamma\delta}\,n^\alpha\,n^\gamma\,\hat e_\lambda\left[L^\alpha,X^\beta\right]\\
&= -\frac{i\,\vartheta^2}{\hbar}\sum_{\alpha,\beta,\gamma,\lambda}\epsilon^{\beta\gamma\delta}\,n^\alpha\,n^\gamma\,\hat e_\lambda\sum_\mu i\,\hbar\,\epsilon^{\alpha\beta\mu}\,X^\mu\\
&= \vartheta^2\sum_{\alpha,\beta,\gamma,\lambda,\mu}\epsilon^{\alpha\beta\mu}\,\epsilon^{\beta\gamma\delta}\,n^\alpha\,n^\gamma\,\hat e_\lambda\,X^\mu\\
&= \vartheta^2\sum_{\alpha,\beta,\gamma,\lambda,\mu}\epsilon^{\beta\mu\alpha}\,\epsilon^{\beta\gamma\delta}\,n^\alpha\,n^\gamma\,\hat e_\lambda\,X^\mu\\
&= \sum_{\alpha,\gamma,\lambda,\mu}\left(\delta^{\mu\gamma}\,\delta^{\alpha\lambda}-\delta^{\mu\lambda}\,\delta^{\alpha\gamma}\right)n^\alpha\,n^\gamma\,\hat e_\lambda\,X^\mu\\
&= \sum_{\alpha,\mu}\left(n^\alpha\,n^\mu\,\hat e_\alpha\,X^\mu-n^\alpha\,n^\alpha\,\hat e_\mu\,X^\mu\right)\\
&= -\vartheta^2\left(\vec X-\left(\vec X\cdot\vec n\right)\vec n\right)
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\vec X\right]\right]\right] &= -\frac{i\,\vartheta^3}{\hbar}\left(\left[L^{\vec n},\vec X\right]-\left[L^{\vec n},\left(\vec X\cdot\vec n\right)\vec n\right]\right)\\
&= -\frac{i\,\vartheta^3}{\hbar}\left(i\,\hbar\,\vec X\times\vec n-\left[\sum_\alpha n^\alpha\,L^\alpha,\sum_{\beta,\gamma}X^\beta\,n^\beta\,n^\gamma\,\hat e_\gamma\right]\right)\\
&= \vartheta^3\,\vec X\times\vec n+\frac{i\,\vartheta^3}{\hbar}\sum_{\alpha,\beta,\gamma}n^\alpha\,n^\beta\,n^\gamma\,\hat e_\gamma\left[L^\alpha,X^\beta\right]\\
&= \vartheta^3\,\vec X\times\vec n+\frac{i\,\vartheta^3}{\hbar}\sum_{\alpha,\beta,\gamma}n^\alpha\,n^\beta\,n^\gamma\,\hat e_\gamma\sum_\lambda i\,\hbar\,\epsilon^{\alpha\beta\lambda}\,X^\lambda\\
&= \vartheta^3\,\vec X\times\vec n-\vartheta^3\sum_{\alpha,\beta,\gamma,\lambda}\epsilon^{\alpha\beta\lambda}\,n^\alpha\,n^\beta\,X^\lambda\,n^\gamma\,\hat e_\gamma\\
&= \vartheta^3\,\vec X\times\vec n-\vartheta^3\left(\left(\vec n\times\vec n\right)\cdot\vec X\right)\vec n\\
&= \vartheta^3\,\vec X\times\vec n
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec n},\vec X\right]\right]\right]\right] &= \frac{i\,\vartheta^4}{\hbar}\left[L^{\vec n},\vec X\times\vec n\right]\\
&= -\vartheta^2\left(-\frac{i\,\vartheta^2}{\hbar}\left[L^{\vec n},\vec X\times\vec n\right]\right)\\
&= \vartheta^4\left(\vec X-\left(\vec X\cdot\vec n\right)\vec n\right)
\end{aligned}
$$

$$\underbrace{\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\cdots\left[\frac{i\,\vartheta}{\hbar}\,L^{\vec{n}},\vec{X}\right]\right]\right]}_{m} = \begin{cases} (-1)^{\frac{m}{2}}\,\vartheta^m\,\left(\vec{X}-\left(\vec{X}\cdot\vec{n}\right)\vec{n}\right), & m \text{ even} \\ \\ (-1)^{\frac{m+1}{2}}\,\vartheta^m\,\vec{X}\times\vec{n}, & m \text{ odd} \end{cases}$$

$$= \begin{cases} (-1)^{\frac{m}{2}}\,\vartheta^m\,\vec{X}^{\perp}, & m \text{ even} \\ \\ (-1)^{\frac{m+1}{2}}\,\vartheta^m\,\vec{X}^{\perp}\times\vec{n}, & m \text{ odd} \end{cases}$$

(since $\vec{X}-\vec{X}^{\parallel}=\vec{X}^{\perp}$ and $\vec{X}\times\vec{n}=\vec{X}^{\perp}\times\vec{n}+\vec{X}^{\parallel}\times\vec{n}=\vec{X}^{\perp}\times\vec{n}$)

$$\begin{aligned} \vec{X}\left(\vec{\vartheta}\right) &= \vec{X}^{\parallel}+\vec{X}^{\perp}+\sum_{\substack{m=2\\ m \text{ even}}}^{\infty}\frac{1}{m!}\,(-1)^{\frac{m}{2}}\,\vartheta^m\,\vec{X}^{\perp}+\sum_{\substack{m=1\\ m \text{ odd}}}^{\infty}\frac{1}{m!}\,(-1)^{\frac{m+1}{2}}\,\vartheta^m\,\vec{X}^{\perp}\times\vec{n} \\ &= \vec{X}^{\parallel}+\sum_{\substack{m=0\\ m \text{ even}}}^{\infty}\frac{1}{m!}(-1)^{\frac{m}{2}}\,\vartheta^m\,\vec{X}^{\perp}+\sum_{\substack{m=1\\ m \text{ odd}}}^{\infty}\frac{1}{m!}(-1)^{\frac{m+1}{2}}\,\vartheta^m\,\vec{X}^{\perp}\times\vec{n} \\ &= \vec{X}^{\parallel}+\vec{X}^{\perp}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\,(-1)^m\,\vartheta^{2m}+\vec{X}^{\perp}\times\vec{n}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\,(-1)^{m+1}\,\vartheta^{2m+1} \\ &= \vec{X}^{\parallel}+\vec{X}^{\perp}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\,(-1)^m\,\vartheta^{2m}-\vec{X}^{\perp}\times\vec{n}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\,(-1)^m\,\vartheta^{2m+1} \\ &= \vec{X}^{\parallel}+\vec{X}^{\perp}\cos\vartheta-\vec{X}^{\perp}\times\vec{n}\sin\vartheta \end{aligned}$$