MAU34403: Quantum Mechanics I
Homework 5 due 21/10/2021

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}\left[H, \hat{A}\right]$$

$$\begin{aligned}
\frac{d\vec{X}_a}{dt} &= \frac{i}{\hbar}\left[H, \vec{X}_a\right] \\
&= \frac{i}{\hbar}\left[\sum_{b=1}^{N}\frac{\vec{P}_b^2}{2m_b} + V\left(\left\{\vec{X}_i\right\}\right), \vec{X}_a\right] \\
&= \sum_{b=1}^{N}\frac{i}{2m_b\,\hbar}\left[\vec{P}_b^2, \vec{X}_a\right] + 0 \\
&= \sum_{b=1}^{N}\frac{i}{2m_b\,\hbar}\left(\vec{P}_b\left[\vec{P}_b, \vec{X}_a\right] + \left[\vec{P}_b, \vec{X}_a\right]\vec{P}_b\right) \\
&= \sum_{b=1}^{N}\frac{1}{2m_b}\left(\vec{P}_b\,\delta^{ab} + \delta^{ab}\vec{P}_b\right) \\
&= \frac{\vec{P}_a}{m_a}
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{P}_a}{dt} &= \frac{i}{\hbar}\left[H, \vec{P}_a\right] \\
&= \frac{i}{\hbar}\left[\sum_{b=1}^{N}\frac{\vec{P}_b^2}{2m_b} + V\left(\left\{\vec{X}_i\right\}\right), -i\hbar\frac{\partial}{\partial\vec{X}_a}\right] \\
&= 0 + \left[V\left(\left\{\vec{X}_i\right\}\right), \frac{\partial}{\partial\vec{X}_a}\right] \\
\frac{d\vec{P}_a}{dt}\, f\left(\vec{X}_j\right) &= \left[V\left(\left\{\vec{X}_i\right\}\right), \frac{\partial}{\partial\vec{X}_a}\right] f\left(\vec{X}_j\right) \\
&= V\,\frac{\partial f}{\partial\vec{X}_a} - \frac{\partial}{\partial\vec{X}_a}(Vf) \\
&= V\,\frac{\partial f}{\partial\vec{X}_a} - \frac{\partial V}{\partial\vec{X}_a}\,f - V\,\frac{\partial f}{\partial\vec{X}_a} \\
&= -\frac{\partial V}{\partial\vec{X}_a}\,f\left(\vec{X}_j\right) \\
\implies \frac{d\vec{P}_a}{dt} &= -\frac{\partial V\left(\left\{\vec{X}_i\right\}\right)}{\partial\vec{X}_a}
\end{aligned}$$

$$\begin{aligned}
\left\langle\frac{d\vec{X}_a}{dt}\right\rangle &= \left\langle\frac{\vec{P}_a}{m_a}\right\rangle &\qquad \left\langle\frac{d\vec{P}_a}{dt}\right\rangle &= \left\langle-\frac{\partial V}{\partial\vec{X}_a}\right\rangle \\
\langle\psi|\,\frac{d\vec{X}_a}{dt}\,|\psi\rangle &= \langle\psi|\,\frac{\vec{P}_a}{m_a}\,|\psi\rangle &\qquad \langle\psi|\,\frac{d\vec{P}_a}{dt}\,|\psi\rangle &= \langle\psi|-\frac{\partial V}{\partial\vec{X}_a}\,|\psi\rangle \\
\frac{d}{dt}\Big(\langle\psi|\,\vec{X}_a\,|\psi\rangle\Big) &= \frac{1}{m_a}\,\langle\psi|\,\vec{P}_a\,|\psi\rangle &\qquad \frac{d}{dt}\Big(\langle\psi|\,\vec{P}_a\,|\psi\rangle\Big) &= -\,\langle\psi|\,\frac{\partial V}{\partial\vec{X}_a}\,|\psi\rangle \quad \text{(since } |\psi\rangle \text{ is time-independent)} \\
\implies \frac{d}{dt}\left\langle\vec{X}_a\right\rangle &= \frac{1}{m_a}\left\langle\vec{P}_a\right\rangle &\qquad \frac{d}{dt}\left\langle\vec{P}_a\right\rangle &= -\left\langle\frac{\partial V}{\partial\vec{X}_a}\right\rangle
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dt}\left\langle\vec{X}_a\right\rangle &= \frac{1}{m_a}\,\frac{d}{dt}\left\langle\vec{P}_a\right\rangle \\
&= \frac{1}{m_a}\left(-\left\langle\frac{\partial V}{\partial\vec{X}_a}\right\rangle\right) \\
\implies \left\langle F\left(\vec{X}_a\right)\right\rangle &= m_a\frac{d^2}{dt}\left\langle\vec{X}_a\right\rangle \qquad \left(\text{where } F \equiv \tfrac{\partial V}{\partial\vec{X}_a}\right)
\end{aligned}$$

This equation is very comparable to, but not the same as, Newton's equation in classical mechanics relating force, mass and acceleration, i.e. $F = ma$. If we had that $\left\langle F\left(\vec{X}_a\right)\right\rangle = F\left(\left\langle\vec{X}_a\right\rangle\right)$, then we would have the same as in classical mechanics. This is only true for certain functions of V; for example, if V was quadratic in $\vec{X}_a$ then its derivative would be linear in $\vec{X}_a$. Since we have that

$$\langle f(x)\rangle = \langle ax + b\rangle = a\langle x\rangle + b = f(\langle x\rangle)\,,$$

then the equation matches its classical counterpart. This is not true for other functions of V, such as those cubic in $\vec{X}_a$, as then for its derivative we have $\langle x^2 \rangle \neq \langle x \rangle^2$, in general.

Problem 2

(a)

$$\begin{aligned}
U(t) &= e^{-\frac{i}{\hbar}tH} \\
&= e^{-\frac{i\gamma B}{\hbar}t\sigma^z} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\gamma B}{2}\, t\, \sigma^z\right)^n \\
&= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{1}{n!}\left(\frac{i\gamma B}{2}\, t\right)^n (\sigma^z)^n + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n!}\left(\frac{i\gamma B}{2}\, t\right)^n (\sigma^z)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\gamma B}{2}\, t\right)^{2n} I + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\gamma B}{2}\, t\right)^{2n+1} \sigma^z \\
&= \cos\left(\frac{\gamma B}{2}\, t\right) I + i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z \\
U^\dagger(t) &= \cos\left(\frac{\gamma B}{2}\, t\right) I - i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z
\end{aligned}$$

$$\begin{aligned}
S^\alpha(t) &= U^\dagger(t)\, S^\alpha\, U(t) \\
&= \left(\cos\left(\frac{\gamma B}{2}\, t\right) I - i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\right) \frac{\hbar}{2}\sigma^\alpha \left(\cos\left(\frac{\gamma B}{2}\, t\right) I + i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\right) \\
&= \frac{\hbar}{2}\left(\cos\left(\frac{\gamma B}{2}\, t\right) \sigma^\alpha - i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^\alpha\right) \left(\cos\left(\frac{\gamma B}{2}\, t\right) I + i \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\right) \\
&= \frac{\hbar}{2}\left(\cos^2\left(\frac{\gamma B}{2}\, t\right) \sigma^\alpha + i \cos\left(\frac{\gamma B}{2}\, t\right) \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^\alpha\sigma^z\right. \\
&\quad \left. - i \sin\left(\frac{\gamma B}{2}\, t\right) \cos\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^\alpha + \sin^2\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^\alpha\sigma^z\right) \\
\alpha \neq z : S^\alpha(t) &= \frac{\hbar}{2}\left(\cos^2\left(\frac{\gamma B}{2}\, t\right) \sigma^\alpha - \sin^2\left(\frac{\gamma B}{2}\, t\right) \sigma^\alpha\sigma^z\sigma^z - 2i \cos\left(\frac{\gamma B}{2}\, t\right) \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^\alpha\right) \\
&\qquad \text{(since } \sigma^i\sigma^j = -\sigma^j\sigma^i\text{)} \\
&= \frac{\hbar}{2}\left(\cos^2\left(\frac{\gamma B}{2}\, t\right) - \sin^2\left(\frac{\gamma B}{2}\, t\right) - 2i \cos\left(\frac{\gamma B}{2}\, t\right) \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\right) \sigma^\alpha \\
&= \frac{\hbar}{2}\left(\cos(\gamma B\, t) - i \sin(\gamma B\, t)\sigma^z\right) \sigma^\alpha \\
\alpha = z : S^z(t) &= \frac{\hbar}{2}\left(\cos^2\left(\frac{\gamma B}{2}\, t\right) \sigma^z + \sin^2\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^z\sigma^z\right. \\
&\quad \left. + i \cos\left(\frac{\gamma B}{2}\, t\right) \sin\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^z - i \sin\left(\frac{\gamma B}{2}\, t\right) \cos\left(\frac{\gamma B}{2}\, t\right) \sigma^z\sigma^z\right) \\
&= \frac{\hbar}{2}\left(\cos^2\left(\frac{\gamma B}{2}\, t\right) + \sin^2\left(\frac{\gamma B}{2}\, t\right)\right) \sigma^z \\
&= \frac{\hbar}{2}\sigma^z
\end{aligned}$$

$$
\begin{aligned}
S^x(t) &= \frac{\hbar}{2}\left(\cos(\gamma B\,t)\,\sigma^x - i\sin(\gamma B\,t)\,\sigma^z\sigma^x\right) \\
&= \frac{\hbar}{2}\left(\cos(\gamma B\,t)\,\sigma^x + \sin(\gamma B\,t)\,\sigma^y\right) \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & \cos(\gamma B\,t) - i\sin(\gamma B\,t) \\ \cos(\gamma B\,t) + i\sin(\gamma B\,t) & 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\left(e^{-i\gamma Bt}\left|\uparrow\right\rangle\left\langle\downarrow\right| + e^{i\gamma Bt}\left|\downarrow\right\rangle\left\langle\uparrow\right|\right) \\
S^y(t) &= \frac{\hbar}{2}\left(\cos(\gamma B\,t)\,\sigma^y - i\sin(\gamma B\,t)\,\sigma^z\sigma^y\right) \\
&= \frac{\hbar}{2}\left(\cos(\gamma B\,t)\,\sigma^y - \sin(\gamma B\,t)\,\sigma^x\right) \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & -i\cos(\gamma B\,t) - \sin(\gamma B\,t) \\ i\cos(\gamma B\,t) - \sin(\gamma B\,t) & 0 \end{pmatrix} \\
&= \frac{i\hbar}{2}\left(-e^{-i\gamma Bt}\left|\uparrow\right\rangle\left\langle\downarrow\right| + e^{i\gamma Bt}\left|\downarrow\right\rangle\left\langle\uparrow\right|\right) \\
S^z(t) &= \frac{\hbar}{2}\,\sigma^z \\
&= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\left|\uparrow\right\rangle\left\langle\uparrow\right| - \left|\downarrow\right\rangle\left\langle\downarrow\right|\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle S^x(t)\rangle_\psi &= \left\langle\psi\right| S^x(t)\left|\psi\right\rangle \\
&= \left(\cos\left(\frac{\theta}{2}\right)\left\langle\uparrow\right| + \sin\left(\frac{\theta}{2}\right)\left\langle\downarrow\right|\right)\frac{\hbar}{2}\left(e^{-i\gamma Bt}\left|\uparrow\right\rangle\left\langle\downarrow\right| + e^{i\gamma Bt}\left|\downarrow\right\rangle\left\langle\uparrow\right|\right)\left(\cos\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle\right) \\
&= \frac{\hbar}{2}\left(e^{-i\gamma Bt}\cos\left(\frac{\theta}{2}\right)\left\langle\downarrow\right| + e^{i\gamma Bt}\sin\left(\frac{\theta}{2}\right)\left\langle\uparrow\right|\right)\left(\cos\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle\right) \\
&= \frac{\hbar}{2}\left(e^{-i\gamma Bt}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + e^{i\gamma Bt}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) \\
&= \frac{\hbar}{2}\left(e^{-i\gamma Bt} + e^{i\gamma Bt}\right)\left(\frac{\sin\theta}{2}\right) \\
&= \frac{\hbar}{2}\cos(\gamma B\,t)\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\langle S^y(t)\rangle_\psi &= \left\langle\psi\right| S^y(t)\left|\psi\right\rangle \\
&= \left(\cos\left(\frac{\theta}{2}\right)\left\langle\uparrow\right| + \sin\left(\frac{\theta}{2}\right)\left\langle\downarrow\right|\right)\frac{i\hbar}{2}\left(-e^{-i\gamma Bt}\left|\uparrow\right\rangle\left\langle\downarrow\right| + e^{i\gamma Bt}\left|\downarrow\right\rangle\left\langle\uparrow\right|\right)\left(\cos\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle\right) \\
&= \frac{i\hbar}{2}\left(-e^{-i\gamma Bt}\cos\left(\frac{\theta}{2}\right)\left\langle\downarrow\right| + e^{i\gamma Bt}\sin\left(\frac{\theta}{2}\right)\left\langle\uparrow\right|\right)\left(\cos\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle\right) \\
&= \frac{i\hbar}{2}\left(-e^{-i\gamma Bt}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + e^{i\gamma Bt}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) \\
&= -\frac{\hbar}{2i}\left(-e^{-i\gamma Bt} + e^{i\gamma Bt}\right)\left(\frac{\sin\theta}{2}\right) \\
&= -\frac{\hbar}{2}\sin(\gamma B\,t)\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\langle S^z(t)\rangle_\psi &= \langle\psi|\, S^z(t)\, |\psi\rangle \\
&= \left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)\langle\downarrow|\right)\frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\right) \\
&= \frac{\hbar}{2}\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| - \sin\left(\frac{\theta}{2}\right)\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\right) \\
&= \frac{\hbar}{2}\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) \\
&= \frac{\hbar}{2}\cos\theta
\end{aligned}
$$

We can interpret the expectation values as the projection of a vector onto each corresponding axis, where the vector has magnitude $\frac{\hbar}{2}$, is at an angle θ to the z-axis, and rotates clockwise around the z-axis with period $\frac{2\pi}{\gamma B}$.

Problem 3

(a)

$$
\begin{aligned}
\psi(p) &= \langle p|\psi\rangle \\
&= \frac{1}{2\pi\,\hbar}\int_{-\infty}^{\infty} dx\, e^{-\frac{ipx}{\hbar}}\,\psi(x) \\
&= \frac{1}{2\pi\,\hbar}\,\frac{1}{\sqrt{\sqrt{\pi}\,\Delta}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{i\,p\,x}{\hbar}\right)\exp\left(\frac{i\,k\,(x-a)}{\hbar} - \frac{(x-a)^2}{2\Delta^2}\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{i\,p\,x}{\hbar} + \frac{i\,k\,x}{\hbar} - \frac{i\,k\,a}{\hbar} - \frac{x^2}{2\Delta^2} - \frac{a^2}{2\Delta^2} + \frac{x\,a}{\Delta^2}\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2}\right)\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2\Delta^2}\,x^2 + \left(\frac{i\,(k-p)}{\hbar} + \frac{a}{\Delta^2}\right)x\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2}\right)\int_{-\infty}^{\infty} dx \exp\left(-\alpha\,x^2 + \beta\,x\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(labelling } \alpha = \tfrac{1}{2\Delta^2} \text{ and } \beta = \tfrac{i(k-p)}{\hbar} + \tfrac{a}{\Delta^2}\text{)} \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2}\right)\int_{-\infty}^{\infty} dx \exp\left(-\alpha\left(x^2 - \frac{\beta}{\alpha}\,x\right)\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2}\right)\int_{-\infty}^{\infty} dx \exp\left(-\alpha\left(\left(x^2 - \frac{\beta}{\alpha}\,x + \frac{\beta^2}{4\alpha^2}\right) - \frac{\beta^2}{4\alpha^2}\right)\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2}\right)\int_{-\infty}^{\infty} dx \exp\left(-\alpha\left(x - \frac{\beta}{2\alpha}\right)^2 + \frac{\beta^2}{4\alpha}\right) \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2} + \frac{\beta^2}{4\alpha}\right)\int_{-\infty}^{\infty} dy \exp\left(-\alpha\,y^2\right) \qquad \text{(labelling } y = x - \tfrac{\beta}{2\alpha}\text{)} \\
&= \frac{1}{\sqrt{2\pi^{\frac{3}{2}}\,\hbar\,\Delta}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2} + \frac{\beta^2}{4\alpha}\right)\sqrt{\frac{\pi}{\alpha}} \\
&= \sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{i\,k\,a}{\hbar} - \frac{a^2}{2\Delta^2} + \frac{\Delta^2}{2}\left(\frac{i\,(k-p)}{\hbar} + \frac{a}{\Delta^2}\right)^2\right) \\
&= \sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{\Delta^2\,(k-p)^2 + 2i\,a\,p\,\hbar}{2\hbar^2}\right) \qquad \text{(simplifying using Mathematica)}
\end{aligned}
$$

$$
\begin{aligned}
\psi(p,t) &= U(t)\,\psi(p) \\
&= \exp\left(-\frac{i\,t}{2m\,\hbar}\,P^2\right)\psi(p) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\,t}{2m\,\hbar}\,P^2\right)^n\psi(p) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\,t}{2m\,\hbar}\right)^n P^{2n}\psi(p) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\,t}{2m\,\hbar}\right)^n p^{2n}\,\psi(p) && (\text{since } P\psi(p) = p\,\psi(p)) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\,t\,p^2}{2m\,\hbar}\right)^n\psi(p) \\
&= \exp\left(-\frac{i\,t\,p^2}{2m\,\hbar}\right)\sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{\Delta^2\,(k-p)^2 + 2i\,a\,p\,\hbar}{2\hbar^2}\right) \\
&= \sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{i\,t\,p^2}{2m\,\hbar} - \frac{\Delta^2\,p^2}{2\hbar^2} + \frac{\Delta^2\,k\,p}{\hbar^2} - \frac{i\,a\,p}{\hbar} - \frac{\Delta^2\,k^2}{2\hbar^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\pi\,\hbar}}\int_{-\infty}^{\infty} dp\, e^{\frac{ipx}{\hbar}}\,\psi(p,t) \\
&= \frac{1}{\sqrt{2\pi\,\hbar}}\int_{-\infty}^{\infty} dp\exp\left(\frac{i\,x\,p}{\hbar}\right)\sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{i\,t\,p^2}{2m\,\hbar} - \frac{\Delta^2\,p^2}{2\hbar^2} + \frac{\Delta^2\,k\,p}{\hbar^2} - \frac{i\,a\,p}{\hbar} - \frac{\Delta^2\,k^2}{2\hbar^2}\right) \\
&= \sqrt{\frac{\Delta}{2\hbar^2\,\pi^{\frac{3}{2}}}}\exp\left(-\frac{\Delta^2\,k^2}{2\hbar^2}\right)\int_{-\infty}^{\infty} dp\exp\left(-\left(\frac{i\,t}{2m\,\hbar} + \frac{\Delta^2}{2\hbar^2}\right)p^2 + \left(\frac{\Delta^2\,k}{\hbar^2} - \frac{i\,a}{\hbar} + \frac{i\,t}{\hbar}\right)p\right) \\
&= \sqrt{\frac{\Delta}{2\hbar^2\,\pi^{\frac{3}{2}}}}\exp\left(-\frac{\Delta^2\,k^2}{2\hbar^2}\right)\int_{-\infty}^{\infty} dp\exp\left(-\gamma\,p^2 + \delta\,p\right) \\
& && \left(\text{labelling } \gamma = \tfrac{i\,t}{2m\,\hbar} + \tfrac{\Delta^2}{2\hbar^2} \text{ and } \delta = \tfrac{\Delta^2\,k}{\hbar^2} - \tfrac{i\,a}{\hbar} + \tfrac{i\,x}{\hbar}\right) \\
&= \sqrt{\frac{\Delta}{2\hbar^2\,\pi^{\frac{3}{2}}}}\exp\left(-\frac{\Delta^2\,k^2}{2\hbar^2} + \frac{\delta^2}{4\gamma}\right)\sqrt{\frac{\pi}{\gamma}} && (\text{as before with } \alpha \to \gamma \text{ and } \beta \to \delta) \\
&= \sqrt{\frac{\Delta}{2\hbar^2\,\sqrt{\pi}}}\sqrt{\frac{1}{\frac{i\,t}{2m\,\hbar} + \frac{\Delta^2}{2\hbar^2}}}\exp\left(-\frac{\Delta^2\,k^2}{2\hbar^2} + \frac{1}{\frac{2i\,t}{m\,\hbar} + \frac{2\Delta^2}{\hbar^2}}\left(\frac{\Delta^2\,k}{\hbar^2} - \frac{i\,a}{\hbar} + \frac{i\,x}{\hbar}\right)^2\right) \\
&= \sqrt{\frac{m\,\Delta}{\sqrt{\pi}\,(m\,\Delta^2 + i\,t\,\hbar)}}\exp\left(\frac{1}{2\hbar^2}\left(-k^2\,\Delta^2 + \frac{m\left(k\,\Delta^2 - i\,(a-x)\,\hbar\right)^2}{m\,\Delta^2 + i\,t\,\hbar}\right)\right) \\
& && (\text{simplifying using Mathematica})
\end{aligned}
$$

$$
\begin{aligned}
|\psi(x,t)|^2 &= \left|\sqrt{\frac{m\,\Delta}{\sqrt{\pi}\,(m\,\Delta^2 + i\,t\,\hbar)}}\exp\left(\frac{1}{2\hbar^2}\left(-k^2\,\Delta^2 + \frac{m\left(k\,\Delta^2 - i\,(a-x)\,\hbar\right)^2}{m\,\Delta^2 + i\,t\,\hbar}\right)\right)\right|^2 \\
&= \left|\sqrt{\frac{m\,\Delta}{\sqrt{\pi}\,(m\,\Delta^2 + i\,t\,\hbar)}}\right|^2\left|\exp\left(\frac{1}{2\hbar^2}\left(-k^2\,\Delta^2 + \frac{m\left(k\,\Delta^2 - i\,(a-x)\,\hbar\right)^2}{m\,\Delta^2 + i\,t\,\hbar}\right)\right)\right|^2 \\
&= \epsilon\cdot\zeta
\end{aligned}
$$

$$\begin{aligned}\epsilon &= \left|\sqrt{\frac{m\,\Delta}{\sqrt{\pi}\,(m\,\Delta^2+i\,t\,\hbar)}}\right|^2\\ &= \frac{m\,\Delta}{\sqrt{\pi}}\,\frac{1}{|m\,\Delta^2+i\,t\,\hbar|}\\ &= \frac{m\,\Delta}{\sqrt{\pi\,(m^2\,\Delta^4+t^2\,\hbar^2)}}\\ &= \frac{m\,\Delta}{\sqrt{\pi\,(m^2\,\Delta^4+t^2\,\hbar^2)}}\end{aligned}$$

$$\begin{aligned}\zeta &= \left|\exp\left(\frac{1}{2\hbar^2}\left(-k^2\,\Delta^2+\frac{m\left(k\,\Delta^2-i\,(a-x)\,\hbar\right)^2}{m\,\Delta^2+i\,t\,\hbar}\right)\right)\right|^2\\ &= \exp\left(2\Re\left(\frac{1}{2\hbar^2}\left(-k^2\,\Delta^2+\frac{m\left(k\,\Delta^2-i\,(a-x)\,\hbar\right)^2}{m\,\Delta^2+i\,t\,\hbar}\right)\right)\right)\\ &= \exp\left(\Re\left(\frac{1}{\hbar^2}\left(-k^2\,\Delta^2+\frac{m\left(k^2\,\Delta^4-2i\,\hbar\,k\,\Delta^2\,(a-x)-\hbar^2\left(a^2-2a\,x+x^2\right)\right)}{m\,\Delta^2+i\,t\,\hbar}\right)\right)\right)\\ &= \exp\left(\Re\left(\frac{-k^2\,\Delta^2\left(m^2\,\Delta^4+t^2\,\hbar^2\right)+m\left(k^2\,\Delta^4-2i\,\hbar\,k\,\Delta^2\,a+2i\,\hbar\,k\,\Delta^2\,x-\hbar^2\,a^2+2\hbar^2\,a\,x-\hbar^2\,x^2\right)\left(m\,\Delta^2-i\,t\,\hbar\right)}{\hbar^2\,(m^2\,\Delta^4+t^2\,\hbar^2)}\right)\right)\\ &= \exp\left(\Re\left(\frac{-m^2\,k^2\,\Delta^6-k^2\,\Delta^2\,t^2\,\hbar^2+m^2\,k^2\,\Delta^6-2m\,\hbar^2\,t\,k\,\Delta^2\,a+2m\,\hbar^2\,t\,k\,\Delta^2\,x-m^2\,\Delta^2\,\hbar^2\,a^2+2m^2\,\Delta^2\,\hbar^2\,a\,x-m^2\,\Delta^2\,\hbar^2\,x^2}{\hbar^2\,(m^2\,\Delta^4+t^2\,\hbar^2)}\right.\right.\\ &\qquad\left.\left.+\;\mathcal{O}(i)\right)\right)\\ &= \exp\left(-\frac{m^2\,\Delta^2}{m^2\,\Delta^4+t^2\,\hbar^2}\,x^2+2\left(\frac{m^2\,\Delta^2\,a+m\,t\,k\,\Delta^2}{m^2\,\Delta^4+t^2\,\hbar^2}\right)x-\frac{k^2\,\Delta^2\,t^2+2m\,t\,k\,\Delta^2\,a+m^2\,\Delta^2\,a^2}{m^2\,\Delta^4+t^2\,\hbar^2}\right)\\ &= \exp\left(-\eta\,x^2+\theta\,x+\lambda\right)\end{aligned}$$

$$\begin{aligned}\implies |\psi(x,t)|^2 &= \epsilon\exp\left(-\eta\,x^2+\theta\,x+\lambda\right)\\ &= \epsilon\exp\left(\lambda+\frac{\theta^2}{4\eta}\right)\exp\left(-\eta\left(x-\frac{\theta}{2\eta}\right)^2\right) \qquad \text{(completing the square as before)}\end{aligned}$$

$$\begin{aligned}\lambda+\frac{\theta^2}{4\eta} &= -\frac{k^2\,\Delta^2\,t^2+2m\,t\,k\,\Delta^2\,a+m^2\,\Delta^2\,a^2}{m^2\,\Delta^4+t^2\,\hbar^2}+\left(\frac{m^2\,\Delta^2\,a+m\,t\,k\,\Delta^2}{m^2\,\Delta^4+t^2\,\hbar^2}\right)^2\left(\frac{m^2\,\Delta^4+t^2\,\hbar^2}{m^2\,\Delta^2}\right)\\ &= -\frac{k^2\,\Delta^2\,t^2+2m\,t\,k\,\Delta^2\,a+m^2\,\Delta^2\,a^2}{m^2\,\Delta^4+t^2\,\hbar^2}+\left(\frac{m^4\,\Delta^4\,a^2+2m^3\,\Delta^4\,a\,t\,k+m^2\,t^2\,k^2\,\Delta^4}{m^2\,\Delta^2\,(m^2\,\Delta^4+t^2\,\hbar^2)}\right)\\ &= \frac{-k^2\,\Delta^2\,t^2-2m\,t\,k\,\Delta^2\,a-m^2\,\Delta^2\,a^2+m^2\,\Delta^2\,a^2+2m\,\Delta^2\,a\,t\,k+t^2\,k^2\,\Delta^2}{m^2\,\Delta^4+t^2\,\hbar^2}\\ &= 0\end{aligned}$$

$$\implies |\psi(x,t)|^2 = \frac{m\,\Delta}{\sqrt{\pi\,(m^2\,\Delta^4+t^2\,\hbar^2)}}\exp\left(-\frac{1}{2}\left(\frac{x-\frac{\theta}{2\eta}}{\frac{1}{\sqrt{2\eta}}}\right)^2\right)$$

This is similar to the form of a normalised Gaussian distribution

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right).$$

If the probability density has the same form as the normalised Gaussian distribution, then we can immediately say that it is normalised. For this to be true, we can equate the coefficients of the exponential and equate the exponentials and if we get the same σ from the coefficient and the exponent, we know that the probability density is a normalised Gaussian distribution.

$$\frac{m\,\Delta}{\sqrt{\pi\left(m^2\,\Delta^4+t^2\,\hbar^2\right)}}=\frac{1}{\sigma\sqrt{2\pi}} \qquad \exp\left(-\frac{1}{2}\left(\frac{x-\frac{\theta}{2\eta}}{\frac{1}{\sqrt{2\eta}}}\right)^2\right)=\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$\sigma\sqrt{2}=\frac{\sqrt{m^2\,\Delta^4+t^2\,\hbar^2}}{m\,\Delta} \qquad \sigma=\frac{1}{\sqrt{2\eta}}$$

$$\sigma=\frac{1}{m\,\Delta}\sqrt{\frac{m^2\,\Delta^4+t^2\,\hbar^2}{2}} \qquad =\sqrt{\frac{m^2\,\Delta^4+t^2\,\hbar^2}{2m^2\,\Delta^2}}$$

$$=\frac{1}{m\,\Delta}\sqrt{\frac{m^2\,\Delta^4+t^2\,\hbar^2}{2}}$$

Thus $|\psi(x,t)|^2$ is a normalised Gaussian function, and therefore normalised.

$$\begin{aligned}
|\psi(p,t)|^2 &= \left|\sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\exp\left(-\frac{i\,t\,p^2}{2m\,\hbar}-\frac{\Delta^2\,p^2}{2\hbar^2}+\frac{\Delta^2\,k\,p}{\hbar^2}-\frac{i\,a\,p}{\hbar}-\frac{\Delta^2\,k^2}{2\hbar^2}\right)\right|^2\\
&= \left|\sqrt{\frac{\Delta}{\hbar\sqrt{\pi}}}\right|^2\left|\exp\left(-\frac{i\,t\,p^2}{2m\,\hbar}-\frac{\Delta^2\,p^2}{2\hbar^2}+\frac{\Delta^2\,k\,p}{\hbar^2}-\frac{i\,a\,p}{\hbar}-\frac{\Delta^2\,k^2}{2\hbar^2}\right)\right|^2\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(2\Re\left(-\frac{i\,t\,p^2}{2m\,\hbar}-\frac{\Delta^2\,p^2}{2\hbar^2}+\frac{\Delta^2\,k\,p}{\hbar^2}-\frac{i\,a\,p}{\hbar}-\frac{\Delta^2\,k^2}{2\hbar^2}\right)\right)\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(-\frac{\Delta^2}{\hbar^2}\,p^2+\frac{2\Delta^2\,k}{\hbar^2}\,p-\frac{\Delta^2\,k^2}{\hbar^2}\right)\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(-\frac{\Delta^2\,k^2}{\hbar^2}\right)\exp\left(-\nu\,p^2+\xi\,p\right)\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(-\frac{\Delta^2\,k^2}{\hbar^2}+\frac{\xi^2}{4\nu}\right)\exp\left(-\nu\left(p-\frac{\xi}{2\nu}\right)^2\right) \qquad \text{(completing the square as before)}\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(-\frac{\Delta^2\,k^2}{\hbar^2}+\frac{\Delta^4\,k^2}{\hbar^4}\,\frac{\hbar^2}{\Delta^2}\right)\exp\left(-\frac{1}{2}\left(\frac{p-\frac{\xi}{2\nu}}{\frac{1}{\sqrt{2\nu}}}\right)^2\right)\\
&= \frac{\Delta}{\hbar\sqrt{\pi}}\exp\left(-\frac{1}{2}\left(\frac{p-\frac{\xi}{2\nu}}{\frac{1}{\sqrt{2\nu}}}\right)^2\right)
\end{aligned}$$

As before, we can check if the coefficient σ is equal the exponent σ then we can say the density is a normalised Gaussian distribution.

$$\frac{\Delta}{\hbar\sqrt{\pi}}=\frac{1}{\sigma\sqrt{2\pi}} \qquad \exp\left(-\frac{1}{2}\left(\frac{p-\frac{\xi}{2\nu}}{\frac{1}{\sqrt{2\nu}}}\right)^2\right)=\exp\left(-\frac{1}{2}\left(\frac{p-\mu}{\sigma}\right)^2\right)$$

$$\sigma\sqrt{2}=\frac{\hbar}{\Delta} \qquad \sigma=\frac{1}{\sqrt{2\nu}}$$

$$\sigma=\frac{\hbar}{\Delta\sqrt{2}} \qquad =\sqrt{\frac{\hbar^2}{2\Delta^2}}$$

$$=\frac{\hbar}{\Delta\sqrt{2}}$$

Thus $|\psi(p,t)|^2$ is a normalised Gaussian function, and therefore normalised.

(b)

Since the expectation value and uncertainty are equivalent to the mean and standard deviation respectively, it is sufficient to find μ and σ for $|\psi(x,t)|^2$ and $|\psi(x,t)|^2$.

$$\begin{aligned} \mu_X &= \frac{\theta}{2\eta} \\ &= \frac{m^2\,\Delta^2\,a + m\,t\,k\,\Delta^2}{m^2\,\Delta^4 + t^2\,\hbar^2}\,\frac{m^2\,\Delta^4 + t^2\,\hbar^2}{m^2\,\Delta^2} \\ &= a + \frac{k\,t}{m} \\ \sigma_X &= \frac{1}{m\,\Delta}\sqrt{\frac{m^2\,\Delta^4 + t^2\,\hbar^2}{2}} \end{aligned} \qquad \begin{aligned} \mu_P &= \frac{\xi}{2\nu} \\ &= \frac{\Delta^2\,k}{\hbar^2}\,\frac{\hbar^2}{\Delta^2} \\ &= k \\ \sigma_P &= \frac{\hbar}{\Delta\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \langle X \rangle_{\psi(x,t)} &= a + \frac{k\,t}{m} \\ \Delta X &= \frac{1}{m\,\Delta}\sqrt{\frac{m^2\,\Delta^4 + t^2\,\hbar^2}{2}} \end{aligned} \qquad \begin{aligned} \langle P \rangle_{\psi(p,t)} &= k \\ \Delta P &= \frac{\hbar}{\Delta\sqrt{2}} \end{aligned}$$

While the uncertainty of P does not depend on time and is thus constant, the uncertainty in X will get larger as time goes on, and thus will not be very small for a substantial period of time. At time $t = 0$ we have

$$\begin{aligned} \Delta X\,\Delta P &= \frac{1}{m\,\Delta}\sqrt{\frac{m^2\,\Delta^4}{2}}\,\frac{\hbar}{\Delta\sqrt{2}} \\ &= \frac{\hbar}{2}, \end{aligned}$$

as expected for $t = 0$. This product of uncertainties will only increase as time goes on, and so Heisenberg's uncertainty principle is satisfied.