MAU34403: Quantum Mechanics I
Homework 4 due 14/10/2021

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Problem 1

$$\mathbb{S}^\pm = \mathbb{S}^1 \pm i\,\mathbb{S}^2 = S^1\otimes I + I\otimes S^1 \pm i\left(S^2\otimes I + I\otimes S^2\right) \qquad \mathbb{S}^3 = S^3\otimes I + I\otimes S^3$$

$$\begin{aligned}
\left[\mathbb{S}^+,\mathbb{S}^-\right] =& \left[S^1\otimes I + I\otimes S^1 + i\left(S^2\otimes I + I\otimes S^2\right), S^1\otimes I + I\otimes S^1 - i\left(S^2\otimes I + I\otimes S^2\right)\right]\\
=& \left[S^1\otimes I + I\otimes S^1, S^1\otimes I + I\otimes S^1\right] + \left[S^1\otimes I + I\otimes S^1, -i\left(S^2\otimes I + I\otimes S^2\right)\right]\\
&+ \left[i\left(S^2\otimes I + I\otimes S^2\right), S^1\otimes I + I\otimes S^1\right] + \left[i\left(S^2\otimes I + I\otimes S^2\right), -i\left(S^2\otimes I + I\otimes S^2\right)\right]\\
=& \,0 + 2i\left[S^2\otimes I + I\otimes S^2, S^1\otimes I + I\otimes S^1\right] + 0\\
=& \,2i\left(\left[S^2\otimes I, S^1\otimes I\right] + \left[S^2\otimes I, I\otimes S^1\right] + \left[I\otimes S^2, S^1\otimes I\right] + \left[I\otimes S^2, I\otimes S^1\right]\right)\\
=& \,2i\left(S^2S^1\otimes I - S^1S^2\otimes I + S^2\otimes S^1 - S^1\otimes S^2 + S^1\otimes S^2 - S^1\otimes S^2 + I\otimes S^2S^1 - I\otimes S^1S^2\right)\\
=& \,2i\left(\left(S^2S^1 - S^1S^2\right)\otimes I + I\otimes\left(S^2S^1 - S^1S^2\right)\right)\\
=& -2i\left(\left[S^1,S^2\right]\otimes I + I\otimes\left[S^1,S^2\right]\right)\\
=& \,2\hbar\left(S^3\otimes I + I\otimes S^3\right)\\
=& \,2\hbar\,\mathbb{S}^3
\end{aligned}$$

$$\begin{aligned}
\left[\mathbb{S}^\pm,\mathbb{S}^3\right] =& \left[S^1\otimes I + I\otimes S^1 \pm i\left(S^2\otimes I + I\otimes S^2\right), S^3\otimes I + I\otimes S^3\right]\\
=& \left[S^1\otimes I + I\otimes S^1, S^3\otimes I + I\otimes S^3\right] + \left[\pm i\left(S^2\otimes I + I\otimes S^2\right), S^3\otimes I + I\otimes S^3\right]\\
=& \left[S^1\otimes I, S^3\otimes I\right] + \left[S^1\otimes I, I\otimes S^3\right] + \left[I\otimes S^1, S^3\otimes I\right] + \left[I\otimes S^1, I\otimes S^3\right]\\
&\pm i\left(\left[S^2\otimes I, S^3\otimes I\right] + \left[S^2\otimes I, I\otimes S^3\right] + \left[I\otimes S^2, S^3\otimes I\right] + \left[I\otimes S^2, I\otimes S^3\right]\right)\\
=& \,S^1S^3\otimes I - S^3S^1\otimes I + S^1\otimes S^3 - S^1\otimes S^3 + S^3\otimes S^1 - S^3\otimes S^1 + I\otimes S^1S^3 - I\otimes S^3S^1\\
&\pm i\left(S^2S^3\otimes I - S^3S^2\otimes I + S^2\otimes S^3 - S^2\otimes S^3 + S^3\otimes S^2 - S^3\otimes S^2 + I\otimes S^2S^3 - I\otimes S^3S^2\right)\\
=& \left(S^1S^3 - S^3S^1\right)\otimes I + I\otimes\left(S^1S^3 - S^3S^1\right) \pm i\left(\left(S^2S^3 - S^3S^2\right)\otimes I + I\otimes\left(S^2S^3 - S^3S^2\right)\right)\\
=& \left[S^1,S^3\right]\otimes I + I\otimes\left[S^1,S^3\right] \pm i\left(\left[S^2,S^3\right]\otimes I + I\otimes\left[S^2,S^3\right]\right)\\
=& -i\hbar\left(S^2\otimes I + I\otimes S^2\right) \mp \hbar\left(S^1\otimes I + I\otimes S^1\right)\\
=& -i\hbar\,\mathbb{S}^2 \mp \hbar\,\mathbb{S}^1\\
=& \mp\hbar\,\mathbb{S}^\pm
\end{aligned}$$

$$\left[\mathbb{S}^+,\mathbb{S}^-\right] = 2\hbar\,\mathbb{S}^3 \qquad \left[\mathbb{S}^+,\mathbb{S}^3\right] = -\hbar\,\mathbb{S}^+ \qquad \left[\mathbb{S}^-,\mathbb{S}^3\right] = \hbar\,\mathbb{S}^-$$

$$\begin{aligned}
\mathbb{S}^\pm &= S^1\otimes I + I\otimes S^1 \pm i\left(S^2\otimes I + I\otimes S^2\right)\\
&= \frac{\hbar}{2}\left(\sigma^1\otimes I + I\otimes\sigma^1 \pm i\left(\sigma^2\otimes I + I\otimes\sigma^2\right)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{2}{\hbar}\mathbb{S}^\pm &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix} \pm \begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} \pm \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\
&= \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} + \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \pm \begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix} \pm \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\\
&= \begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix} \pm \begin{pmatrix}0&1&1&0\\-1&0&0&1\\-1&0&0&1\\0&-1&-1&0\end{pmatrix}
\end{aligned}$$

$$\mathbb{S}^+ = \hbar \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \hbar \left(|1\rangle \langle 2| + |1\rangle \langle 3| + |2\rangle \langle 4| + |3\rangle \langle 4| \right)$$

$$\mathbb{S}^- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \hbar \left(|2\rangle \langle 1| + |3\rangle \langle 1| + |4\rangle \langle 2| + |4\rangle \langle 3| \right)$$

$$|e_1\rangle = |\uparrow\uparrow\rangle = |1\rangle \qquad |e_0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) = \frac{1}{\sqrt{2}} \left(|2\rangle + |3\rangle \right) \qquad |e_{-1}\rangle = |\downarrow\downarrow\rangle = |4\rangle$$

$$\mathbb{S}^+ |e_1\rangle = \hbar (0 + 0 + 0 + 0) = 0 \qquad \mathbb{S}^+ |e_0\rangle = \frac{\hbar}{\sqrt{2}} \left(|1\rangle + |1\rangle + 0 + 0 \right) = \hbar\sqrt{2} \, |e_1\rangle \qquad \mathbb{S}^+ |e_{-1}\rangle = \hbar \left(0 + 0 + |2\rangle + |3\rangle \right) = \hbar\sqrt{2} \, |e_0\rangle$$

$$\mathbb{S}^- |e_1\rangle = \hbar \left(|2\rangle + |3\rangle + 0 + 0 \right) = \hbar\sqrt{2} \, |e_0\rangle \qquad \mathbb{S}^- |e_0\rangle = \frac{\hbar}{\sqrt{2}} \left(0 + 0 + |4\rangle + |4\rangle \right) = \hbar\sqrt{2} \, |e_{-1}\rangle \qquad \mathbb{S}^- |e_{-1}\rangle = \hbar (0 + 0 + 0 + 0) = 0$$

$$\mathbb{S}^\pm |e_{\pm 1}\rangle = 0 \qquad \mathbb{S}^\pm |e_{\mp 1}\rangle = \hbar\sqrt{2} \, |e_0\rangle \qquad \mathbb{S}^\pm |e_0\rangle = \hbar\sqrt{2} \, |e_{\pm 1}\rangle$$

The action of the introduced spin operators $\mathbb{S}^\pm$ on the vectors $|e_m\rangle$ can be interpreted as a scaling of $\hbar\sqrt{2}$ and adding to or subtracting from the index m of the vector respectively, where we consider $|e_2\rangle = |e_{-2}\rangle = 0$.

Problem 2

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{10}} \left(2\,|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + 2\,|\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) \\ &= \frac{1}{\sqrt{10}} \left(2\,|e_1\rangle + |e_{-1}\rangle + \frac{1}{\sqrt{2}} |e_0\rangle - \frac{3}{\sqrt{2}} |f\rangle \right) \end{aligned}$$

(a)

After a measurement results in s_0, we know the system must be in a state of a combination of $|e_0\rangle$ and $|f\rangle$, as these eigenvectors correspond to the eigenvalue s_0. Thus the new state $|\phi\rangle$ is

$$|\phi\rangle = \frac{|e_0\rangle \langle e_0|\psi\rangle + |f\rangle \langle f|\psi\rangle}{\sqrt{|\langle e_0|\psi\rangle|^2 + |\langle f|\psi\rangle|^2}}$$

$$\langle e_0|\psi\rangle = \frac{1}{\sqrt{20}} \implies |\langle e_0|\psi\rangle|^2 = \frac{1}{20} \qquad \langle f|\psi\rangle = -\frac{3}{\sqrt{20}} \implies |\langle f|\psi\rangle|^2 = \frac{9}{20}$$

$$\begin{aligned} |\phi\rangle &= \frac{\frac{1}{\sqrt{20}} |e_0\rangle - \frac{3}{\sqrt{20}} |f\rangle}{\sqrt{\frac{1}{20} + \frac{9}{20}}} \\ &= \frac{1}{\sqrt{10}} |e_0\rangle - \frac{3}{\sqrt{10}} |f\rangle \end{aligned}$$

$$|\phi\rangle = \frac{1}{\sqrt{5}} \left(-\,|\uparrow\downarrow\rangle + 2\,|\downarrow\uparrow\rangle \right)$$

(b)

$$|\psi\rangle \xrightarrow{E_1} |\alpha\rangle \xrightarrow{s_0} |\beta\rangle$$
$$E_1 \implies |e_1\rangle\,,\, |e_0\rangle\,,\, |e_{-1}\rangle$$

$$|\alpha\rangle = \frac{|e_1\rangle\langle e_1|\psi\rangle + |e_0\rangle\langle e_0|\psi\rangle + |e_{-1}\rangle\langle e_{-1}|\psi\rangle}{\sqrt{|\langle e_1|\psi\rangle|^2 + |\langle e_0|\psi\rangle|^2 + |\langle e_{-1}|\psi\rangle|^2}}$$

$$\langle e_1|\psi\rangle = \frac{2}{\sqrt{10}} \qquad \langle e_0|\psi\rangle = \frac{1}{\sqrt{20}} \qquad \langle e_{-1}|\psi\rangle = \frac{1}{\sqrt{10}}$$
$$\implies |\langle e_1|\psi\rangle|^2 = \frac{2}{5} \qquad \implies |\langle e_0|\psi\rangle|^2 = \frac{1}{20} \qquad \implies \implies |\langle e_{-1}|\psi\rangle|^2 = \frac{1}{10}$$

$$\begin{aligned}\implies |\alpha\rangle &= \frac{\frac{2}{\sqrt{10}}|e_1\rangle + \frac{1}{\sqrt{20}}|e_0\rangle + \frac{1}{\sqrt{10}}|e_{-1}\rangle}{\sqrt{\frac{2}{5}+\frac{1}{20}+\frac{1}{10}}} \\ &= \sqrt{\frac{8}{11}}|e_1\rangle + \sqrt{\frac{1}{11}}|e_0\rangle + \sqrt{\frac{2}{11}}|e_{-1}\rangle\end{aligned}$$
$$s_0 \implies |e_0\rangle\,,\, |f\rangle$$

$$\langle e_0|\alpha\rangle = \sqrt{\frac{1}{11}} \qquad \langle f|\alpha\rangle = 0$$
$$\implies |\langle e_0|\alpha\rangle|^2 = \frac{1}{11} \qquad |\langle f|\alpha\rangle|^2 = 0$$

$$\begin{aligned}P_1(E_1)\,P_2(s_0) &= \left(|\langle e_1|\psi\rangle|^2 + |\langle e_0|\psi\rangle|^2 + |\langle e_{-1}|\psi\rangle|^2\right)\left(|\langle e_0|\alpha\rangle|^2 + |\langle f|\alpha\rangle|^2\right) \\ &= \left(\frac{2}{5}+\frac{1}{20}+\frac{1}{10}\right)\left(\frac{1}{11}+0\right)\end{aligned}$$
$$P_1(E_1)\,P_2(s_0) = \frac{1}{20}$$

$$|\psi\rangle \xrightarrow{s_0} |\phi\rangle \xrightarrow{E_1} |\gamma\rangle$$
$$|\phi\rangle = \frac{1}{\sqrt{10}}|e_0\rangle - \frac{3}{\sqrt{10}}|f\rangle$$
$$E_1 \implies |e_1\rangle\,,\, |e_0\rangle\,,\, |e_{-1}\rangle$$

$$\langle e_1|\phi\rangle = \langle e_{-1}|\phi\rangle = 0 \qquad \langle e_0|\phi\rangle = \frac{1}{\sqrt{10}}$$
$$\implies |\langle e_1|\phi\rangle|^2 = |\langle e_{-1}|\phi\rangle|^2 = 0 \qquad \implies |\langle e_0|\phi\rangle|^2 = \frac{1}{10}$$

$$\begin{aligned}P_1(s_0)\,P_2(E_1) &= \left(|\langle e_0|\psi\rangle|^2 + |\langle f|\psi\rangle|^2\right)\left(|\langle e_1|\phi\rangle|^2 + |\langle e_0|\phi\rangle|^2 + |\langle e_{-1}|\phi\rangle|^2\right) \\ &= \left(\frac{1}{20}+\frac{9}{20}\right)\left(0+\frac{1}{10}+0\right)\end{aligned}$$
$$P_1(s_0)\,P_2(E_1) = \frac{1}{20}$$

The probabilities are equal since $\mathbb{S}^3$ and H are compatible, and so the order in which measurements are made does not affect the probability of the combination of measurements.

Problem 3

(i)

Getting $\mathbb{S}^3$, H and $|\psi\rangle$ in a usable form

$$\begin{aligned}
\mathbb{S}^3 &= S^3 \otimes I + I \otimes S^3 \\
&= \frac{\hbar}{2}\left(\sigma^3 \otimes I + I \otimes \sigma^3\right) && \text{(tensor product notation)} \\
&= \frac{\hbar}{2}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right) \\
&= \hbar\left(|1\rangle\langle 1| - |4\rangle\langle 4|\right) && \text{(bra-ket notation)}
\end{aligned}$$

$$\begin{aligned}
H &= \frac{3J}{4} + \frac{J}{\hbar^2}\sum_{\alpha=1}^{3} S_1^\alpha S_2^\alpha \\
&= \frac{3J}{4} + \frac{J}{\hbar^2}\sum_{\alpha=1}^{3} \left(S^\alpha \otimes I\right)\left(I \otimes S^\alpha\right) \\
&= \frac{3J}{4} + \frac{J}{4}\sum_{\alpha=1}^{3} \left(\sigma^\alpha \otimes \sigma^\alpha\right) \\
&= \frac{J}{4}\left(3 + \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha\right) && \text{(tensor product notation)} \\
&= \frac{J}{4}\left(3I + \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3\right) \\
&= \frac{J}{4}\left(\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \\
&= \frac{J}{2}\left(2|1\rangle\langle 1| + |2\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 3| + 2|4\rangle\langle 4|\right) && \text{(bra-ket notation)}
\end{aligned}$$

$$|\uparrow\uparrow\rangle = |1\rangle \qquad |\uparrow\downarrow\rangle = |2\rangle \qquad |\downarrow\uparrow\rangle = |3\rangle \qquad |\downarrow\downarrow\rangle = |4\rangle$$

$$\implies |\psi\rangle = \frac{1}{\sqrt{10}}\left(2|1\rangle - |2\rangle + 2|3\rangle + |4\rangle\right)$$

Finding the anti-commutator

$$\begin{aligned}
\left[\mathbb{S}^3, H\right]_+ &= 2\,\mathbb{S}^3 H && (\mathbb{S}^3 \text{ and } H \text{ commute} \implies \mathbb{S}^3 H = H\mathbb{S}^3) \\
&= 2\,\frac{\hbar}{2}\left(\sigma^3 \otimes I + I \otimes \sigma^3\right)\frac{J}{4}\left(3 + \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha\right) && (\text{using tensor product notation}) \\
&= \frac{J\hbar}{4}\left(3\left(\sigma^3 \otimes I + I \otimes \sigma^3\right) + \sum_{\alpha=1}^{3}\left(\sigma^3\sigma^\alpha \otimes \sigma^\alpha + \sigma^\alpha \otimes \sigma^3\sigma^\alpha\right)\right) \\
&= \frac{J\hbar}{4}\left(3\left(\sigma^3 \otimes I + I \otimes \sigma^3\right) + \sigma^3\sigma^1 \otimes \sigma^1 + \sigma^1 \otimes \sigma^3\sigma^1 + \sigma^3\sigma^2 \otimes \sigma^2 + \sigma^2 \otimes \sigma^3\sigma^2 + \sigma^3\sigma^3 \otimes \sigma^3 + \sigma^3 \otimes \sigma^3\sigma^3\right) \\
&= \frac{J\hbar}{4}\left(3\left(\sigma^3 \otimes I + I \otimes \sigma^3\right) + i\,\sigma^2 \otimes \sigma^1 + \sigma^1 \otimes i\,\sigma^2 - i\,\sigma^1 \otimes \sigma^2 - \sigma^2 \otimes i\,\sigma^1 + I \otimes \sigma^3 + \sigma^3 \otimes I\right) \\
&= \frac{J\hbar}{4}\left(4\left(\sigma^3 \otimes I + I \otimes \sigma^3\right) + 0 + 0\right) \\
&= 2J\left(S^3 \otimes I + I \otimes S^3\right) \\
&= 2J\,\mathbb{S}^3
\end{aligned}$$

Finding the expected values

$$\begin{aligned}
\langle \mathbb{S}^3 \rangle_\psi &= \langle \psi |\, \mathbb{S}^3 \,| \psi \rangle \\
&= \frac{1}{\sqrt{10}}\left(2\langle 1| - \langle 2| + 2\langle 3| + \langle 4|\right)\hbar\left(|1\rangle\langle 1| - |4\rangle\langle 4|\right)|\psi\rangle && (\text{using bra-ket notation}) \\
&= \frac{\hbar}{\sqrt{10}}\left(2\langle 1| - \langle 4|\right)\frac{1}{\sqrt{10}}\left(2|1\rangle - |2\rangle + 2|3\rangle + |4\rangle\right) \\
&= \frac{\hbar}{10}(4-1) \\
&= \frac{3\hbar}{10}
\end{aligned}$$

$$\begin{aligned}
\langle H \rangle_\psi &= \langle \psi |\, H \,| \psi \rangle \\
&= \frac{1}{\sqrt{10}}\left(2\langle 1| - \langle 2| + 2\langle 3| + \langle 4|\right)\frac{J}{2}\left(2|1\rangle\langle 1| + |2\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 2| + |3\rangle\langle 3| + 2|4\rangle\langle 4|\right)|\psi\rangle \\
& && (\text{using bra-ket notation}) \\
&= \frac{J}{2\sqrt{10}}\left(4\langle 1| - \langle 2| - \langle 3| + 2\langle 2| + 2\langle 3| + 2\langle 4|\right)\frac{1}{\sqrt{10}}\left(2|1\rangle - |2\rangle + 2|3\rangle + |4\rangle\right) \\
&= \frac{J}{20}(8 - 1 + 2 + 2) \\
&= \frac{11J}{20}
\end{aligned}$$

Finding the uncertainties

$$\begin{aligned}
\Delta\left(\mathbb{S}^3\right)^2 &= \langle s | s \rangle && (\text{where } |s\rangle \equiv \left(\mathbb{S}^3 - \langle \mathbb{S}^3 \rangle\right)|\psi\rangle) \\
&= \langle \psi | \left(\mathbb{S}^3 - \langle \mathbb{S}^3 \rangle\right)\left(\mathbb{S}^3 - \langle \mathbb{S}^3 \rangle\right)|\psi\rangle && (\text{since } \left(\mathbb{S}^3\right)^\dagger = \mathbb{S}^3) \\
&= \langle \psi |\, \mathbb{S}^3\,\mathbb{S}^3 \,|\psi\rangle - 2\langle \mathbb{S}^3 \rangle \langle \psi |\, \mathbb{S}^3 \,|\psi\rangle + \langle \mathbb{S}^3 \rangle^2 \langle \psi | \psi \rangle \\
&= \frac{\hbar}{\sqrt{10}}\left(2\langle 1| - \langle 4|\right)\frac{\hbar}{\sqrt{10}}\left(2|1\rangle - |4\rangle\right) - 2\langle \mathbb{S}^3 \rangle^2 + \langle \mathbb{S}^3 \rangle^2 \\
&= \frac{\hbar^2}{10}(4+1) - \frac{9\hbar^2}{100} \\
&= \frac{41\hbar^2}{100}
\end{aligned}$$

$$
\begin{aligned}
\Delta H^2 &= \langle h|h\rangle && \text{(where } |h\rangle \equiv (H-\langle H\rangle)\,|\psi\rangle\text{)}\\
&= \langle\psi|\,(H-\langle H\rangle)\,(H-\langle H\rangle)\,|\psi\rangle && \text{(since } H^\dagger = H\text{)}\\
&= \langle\psi|\,H\,H\,|\psi\rangle - 2\langle H\rangle\,\langle\psi|\,H\,|\psi\rangle + \langle H\rangle^2\,\langle\psi|\psi\rangle\\
&= \frac{J}{2\sqrt{10}}\,(4\,\langle 1| + \langle 2| + \langle 3| + 2\,\langle 4|)\,\frac{J}{2\sqrt{10}}\,(4\,|1\rangle + |2\rangle + |3\rangle + 2\,|4\rangle) - 2\langle H\rangle^2 + \langle H\rangle^2\\
&= \frac{J^2}{40}\,(16+1+1+4) - \frac{169J^2}{400}\\
&= \frac{99J^2}{400}
\end{aligned}
$$

Checking the relation

$$
\begin{aligned}
\Delta\big(\mathbb{S}^3\big)^2\,\Delta H^2 &= \frac{41\hbar^2}{100}\,\frac{99J^2}{400} & \left(\frac{1}{2}\langle[\mathbb{S}^3,H]_+\rangle - \langle\mathbb{S}^3\rangle\langle H\rangle\right)^2 - \frac{1}{4}\langle[\mathbb{S}^3,H]\rangle^2 &= \left(\frac{1}{2}\langle 2J\,\mathbb{S}^3\rangle - \frac{3\hbar}{10}\,\frac{11J}{20}\right)^2 - \frac{1}{4}\langle 0\rangle^2\\
&= \frac{4059J^2\hbar^2}{40000} & &= \frac{729J^2\hbar^2}{40000}
\end{aligned}
$$

$$
\implies \Delta\big(\mathbb{S}^3\big)^2\,\Delta H^2 > \left(\frac{1}{2}\langle[\mathbb{S}^3,H]_+\rangle - \langle\mathbb{S}^3\rangle\langle H\rangle\right)^2 - \frac{1}{4}\langle[\mathbb{S}^3,H]\rangle^2
$$

(ii)

Getting $\mathbb{S}^1$ in a usable form

$$
\begin{aligned}
\mathbb{S}^1 &= S^1\otimes I + I\otimes S^1\\
&= \frac{\hbar}{2}\left(\sigma^1\otimes I + I\otimes\sigma^1\right) && \text{(tensor product notation)}\\
&= \frac{\hbar}{2}\left(\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\right)\\
&= \frac{\hbar}{2}\left(|1\rangle\langle 2| + |1\rangle\langle 3| + |2\rangle\langle 1| + |2\rangle\langle 4| + |3\rangle\langle 1| + |3\rangle\langle 4| + |4\rangle\langle 2| + |4\rangle\langle 3|\right) && \text{(bra-ket notation)}
\end{aligned}
$$

Finding the commutator

$$
\begin{aligned}
\big[\mathbb{S}^3,\mathbb{S}^1\big] &= \frac{\hbar^2}{4}\left(\left(\sigma^3\otimes I + I\otimes\sigma^3\right)\left(\sigma^1\otimes I + I\otimes\sigma^1\right) - \left(\sigma^1\otimes I + I\otimes\sigma^1\right)\left(\sigma^3\otimes I + I\otimes\sigma^3\right)\right)\\
&= \frac{\hbar^2}{4}\left(\sigma^3\sigma^1\otimes I + \sigma^3\otimes\sigma^1 + \sigma^1\otimes\sigma^3 + I\otimes\sigma^3\sigma^1 - \sigma^1\sigma^3\otimes I - \sigma^1\otimes\sigma^3 - \sigma^3\otimes\sigma^1 - I\otimes\sigma^1\sigma^3\right)\\
&= \frac{\hbar^2}{4}\left(i\,\sigma^2\otimes I + I\otimes i\,\sigma^2 + i\sigma^2\otimes I + I\otimes i\,\sigma^2\right)\\
&= \frac{i\hbar^2}{2}\left(\sigma^2\otimes I + I\otimes\sigma^2\right)\\
&= \frac{i\hbar^2}{2}\left(\begin{pmatrix} 0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}\right)\\
&= \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & -1 & 0\end{pmatrix}\\
&= \frac{\hbar^2}{2}\left(|1\rangle\langle 2| + |1\rangle\langle 3| - |2\rangle\langle 1| + |2\rangle\langle 4| - |3\rangle\langle 1| + |3\rangle\langle 4| - |4\rangle\langle 2| - |4\rangle\langle 3|\right)
\end{aligned}
$$

Finding the anti-commutator

$$
\begin{aligned}
\left[\mathbb{S}^3, \mathbb{S}^1\right]_+ &= \frac{\hbar^2}{4}\left(\left(\sigma^3 \otimes I + I \otimes \sigma^3\right)\left(\sigma^1 \otimes I + I \otimes \sigma^1\right) + \left(\sigma^1 \otimes I + I \otimes \sigma^1\right)\left(\sigma^3 \otimes I + I \otimes \sigma^3\right)\right) \\
&= \frac{\hbar^2}{4}\left(\sigma^3\sigma^1 \otimes I + \sigma^3 \otimes \sigma^1 + \sigma^1 \otimes \sigma^3 + I \otimes \sigma^3\sigma^1 + \sigma^1\sigma^3 \otimes I + \sigma^1 \otimes \sigma^3 + \sigma^3 \otimes \sigma^1 + I \otimes \sigma^1\sigma^3\right) \\
&= \frac{\hbar^2}{4}\left(i\,\sigma^2 \otimes I + I \otimes i\,\sigma^2 - i\,\sigma^2 \otimes I - I \otimes i\,\sigma^2 + 2\sigma^3 \otimes \sigma^1 + 2\sigma^1 \otimes \sigma^3\right) \\
&= \frac{\hbar^2}{2}\left(\sigma^3 \otimes \sigma^1 + \sigma^1 \otimes \sigma^3\right) \\
&= \frac{\hbar^2}{2}\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right) \\
&= \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{2}\left(|1\rangle\langle 2| + |1\rangle\langle 3| + |2\rangle\langle 1| - |2\rangle\langle 4| + |3\rangle\langle 1| - |3\rangle\langle 4| - |4\rangle\langle 2| - |4\rangle\langle 3|\right)
\end{aligned}
$$

Finding the expected values

$$
\begin{aligned}
\langle \mathbb{S}^1 \rangle_\psi &= \langle \psi | \mathbb{S}^1 | \psi \rangle \\
&= \frac{1}{\sqrt{10}}\left(2\langle 1| - \langle 2| + 2\langle 3| + \langle 4|\right)\frac{\hbar}{2}\left(|1\rangle\langle 2| + |1\rangle\langle 3| + |2\rangle\langle 1| + |2\rangle\langle 4| + |3\rangle\langle 1| + |3\rangle\langle 4| + |4\rangle\langle 2| + |4\rangle\langle 3|\right)|\psi\rangle \\
&= \frac{\hbar}{2\sqrt{10}}\left(2\langle 2| + 2\langle 3| - \langle 1| - \langle 4| + 2\langle 1| + 2\langle 4| + \langle 2| + \langle 3|\right)|\psi\rangle \\
&= \frac{\hbar}{2\sqrt{10}}\left(\langle 1| + 3\langle 2| + 3\langle 3| + \langle 4|\right)\frac{1}{\sqrt{10}}\left(2|1\rangle - |2\rangle + 2|3\rangle + |4\rangle\right) \\
&= \frac{\hbar}{20}\left(2 - 3 + 6 + 1\right) \\
&= \frac{3\hbar}{10}
\end{aligned}
$$

$$
\begin{aligned}
\langle \left[\mathbb{S}^3, \mathbb{S}^1\right] \rangle_\psi &= \langle \psi | \left[\mathbb{S}^3, \mathbb{S}^1\right] | \psi \rangle \\
&= \frac{1}{\sqrt{10}}\left(2\langle 1| - \langle 2| + 2\langle 3| + \langle 4|\right)\frac{\hbar^2}{2}\left(|1\rangle\langle 2| + |1\rangle\langle 3| - |2\rangle\langle 1| + |2\rangle\langle 4| - |3\rangle\langle 1| + |3\rangle\langle 4| - |4\rangle\langle 2| - |4\rangle\langle 3|\right)|\psi\rangle \\
&= \frac{\hbar^2}{2\sqrt{10}}\left(2\langle 2| + 2\langle 3| + \langle 1| - \langle 4| - 2\langle 1| + 2\langle 4| - \langle 2| - \langle 3|\right)|\psi\rangle \\
&= \frac{\hbar^2}{2\sqrt{10}}\left(-\langle 1| + \langle 2| + \langle 3| + \langle 4|\right)\frac{1}{\sqrt{10}}\left(2|1\rangle - |2\rangle + 2|3\rangle + |4\rangle\right) \\
&= \frac{\hbar^2}{20}\left(-2 - 1 + 2 + 1\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle [\mathbb{S}^3, \mathbb{S}^1]_+ \rangle &= \langle \psi | \, [\mathbb{S}^3, \mathbb{S}^1]_+ \, |\psi\rangle \\
&= \frac{1}{\sqrt{10}} \left(2\,\langle 1| - \langle 2| + 2\,\langle 3| + \langle 4| \right) \frac{\hbar^2}{2} \left(|1\rangle\,\langle 2| + |1\rangle\,\langle 3| + |2\rangle\,\langle 1| - |2\rangle\,\langle 4| + |3\rangle\,\langle 1| - |3\rangle\,\langle 4| - |4\rangle\,\langle 2| - |4\rangle\,\langle 3| \right) |\psi\rangle \\
&= \frac{\hbar^2}{2\sqrt{10}} \left(2\,\langle 2| + 2\,\langle 3| - \langle 1| + \langle 4| + 2\,\langle 1| - 2\,\langle 4| - \langle 2| - \langle 3| \right) |\psi\rangle \\
&= \frac{\hbar^2}{2\sqrt{10}} \left(\langle 1| + \langle 2| + \langle 3| - \langle 4| \right) \frac{1}{\sqrt{10}} \left(2\,|1\rangle - |2\rangle + 2\,|3\rangle + |4\rangle \right) \\
&= \frac{\hbar^2}{20} \left(2 - 1 + 2 - 1 \right) \\
&= \frac{\hbar^2}{10}
\end{aligned}
$$

Finding the uncertainties

$$
\begin{aligned}
\Delta\left(\mathbb{S}^1\right)^2 &= \langle s | s \rangle && \left(\text{where } |s\rangle \equiv \left(\mathbb{S}^1 - \langle \mathbb{S}^1 \rangle\right) |\psi\rangle\right) \\
&= \langle \psi | \left(\mathbb{S}^1 - \langle \mathbb{S}^1 \rangle\right) \left(\mathbb{S}^1 - \langle \mathbb{S}^1 \rangle\right) |\psi\rangle && \left(\text{since } \left(\mathbb{S}^1\right)^\dagger = \mathbb{S}^3\right) \\
&= \langle \psi | \, \mathbb{S}^1 \, \mathbb{S}^1 \, |\psi\rangle - 2\langle \mathbb{S}^1 \rangle \, \langle \psi | \, \mathbb{S}^1 \, |\psi\rangle + \langle \mathbb{S}^1 \rangle^2 \, \langle \psi | \psi \rangle \\
&= \frac{\hbar}{2\sqrt{10}} \left(\langle 1| + 3\,\langle 2| + 3\,\langle 3| + \langle 4| \right) \frac{\hbar}{2\sqrt{10}} \left(|1\rangle + 3\,|2\rangle + 3\,|3\rangle + |4\rangle \right) - 2\langle \mathbb{S}^1 \rangle^2 + \langle \mathbb{S}^1 \rangle^2 \\
&= \frac{\hbar^2}{40} \left(1 + 9 + 9 + 1 \right) - \frac{9\hbar^2}{100} \\
&= \frac{41\hbar^2}{100}
\end{aligned}
$$

Checking the relation

$$
\begin{aligned}
\Delta\left(\mathbb{S}^3\right)^2 \Delta\left(\mathbb{S}^1\right)^2 &= \frac{41\hbar^2}{100} \, \frac{41\hbar^2}{100} \\
&= \frac{1681\hbar^4}{10000}
\end{aligned}
\qquad
\begin{aligned}
\left(\frac{1}{2} \langle [\mathbb{S}^3, H]_+ \rangle - \langle \mathbb{S}^3 \rangle \langle H \rangle \right)^2 - \frac{1}{4} \langle [\mathbb{S}^3, H] \rangle^2 &= \left(\frac{1}{2} \, \frac{\hbar^2}{10} - \frac{3\hbar}{10} \, \frac{3\hbar}{10} \right)^2 - \frac{0}{4} \\
&= \frac{\hbar^4}{625}
\end{aligned}
$$

$$
\implies \Delta\left(\mathbb{S}^3\right)^2 \Delta H^2 > \left(\frac{1}{2} \langle [\mathbb{S}^3, H]_+ \rangle - \langle \mathbb{S}^3 \rangle \langle H \rangle \right)^2 - \frac{1}{4} \langle [\mathbb{S}^3, H] \rangle^2
$$