MAU34403: Quantum Mechanics I
Homework 3 due 07/10/2021

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Problem 1

(a)

$$\begin{aligned}[(\rho\otimes\sigma)(\mathcal{E}_i),(\rho\otimes\sigma)(\mathcal{E}_j)] &= [\rho(\mathcal{E}_i)\otimes I_{\mathscr{W}}+I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i),\rho(\mathcal{E}_j)\otimes I_{\mathscr{W}}+I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_j)]\\
&= [\rho(\mathcal{E}_i)\otimes I_{\mathscr{W}},\rho(\mathcal{E}_j)\otimes I_{\mathscr{W}}]+[\rho(\mathcal{E}_i)\otimes I_{\mathscr{W}},I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_j)]\\
&\quad +[I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i),\rho(\mathcal{E}_j)\otimes I_{\mathscr{W}}]+[I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_i),I_{\mathscr{V}}\otimes\sigma(\mathcal{E}_j)]\\
&= [\rho(\mathcal{E}_i),\rho(\mathcal{E}_j)]\otimes I_{\mathscr{W}}+0+0+I_{\mathscr{V}}\otimes[\sigma(\mathcal{E}_i),\sigma(\mathcal{E}_j)]\\
&= \rho([\mathcal{E}_i,\mathcal{E}_j])\otimes I_{\mathscr{W}}+I_{\mathscr{V}}\otimes\sigma([\mathcal{E}_i,\mathcal{E}_j]),\end{aligned}$$

thus the Lie algebra commutator is preserved, and so $\rho\otimes\sigma$ is a representation.

(b)

$$(\rho_\beta\otimes\rho_\alpha)(\mathcal{E}_i)\equiv\rho_\beta(\mathcal{E}_i)\otimes I_{\mathscr{V}_\alpha}+I_{\mathscr{V}_\beta}\otimes\rho_\alpha(\mathcal{E}_i)$$

$$\begin{aligned}(\rho_2\otimes\rho_1)(\mathcal{E}_i) &= \rho_2(\mathcal{E}_i)\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i)\\
&\equiv \varrho_{2,1}(\mathcal{E}_i),\ \text{where } (\varrho_{2,1},\mathscr{W}_{2,1})\equiv(\rho_2\otimes\rho_1,\mathscr{V}_2\otimes\mathscr{V}_1)\text{ is a representation of }\mathcal{U}(\mathcal{G})\end{aligned}$$

$$\begin{aligned}(\rho_3\otimes\rho_2\otimes\rho_1)(\mathcal{E}_i) &= (\rho_3\otimes\varrho_{2,1})(\mathcal{E}_i)\\
&= \rho_3(\mathcal{E}_i)\otimes I_{\mathscr{W}_{2,1}}+I_{\mathscr{V}_3}\otimes\varrho_{2,1}(\mathcal{E}_i)\\
&= \rho_3(\mathcal{E}_i)\otimes I_{\mathscr{V}_2\otimes\mathscr{V}_1}+I_{\mathscr{V}_3}\otimes(\rho_2(\mathcal{E}_i)\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i))\\
&= \rho_3(\mathcal{E}_i)\otimes I_{\mathscr{V}_2}\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_3}\otimes\rho_2(\mathcal{E}_i)\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_3}\otimes I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i)\\
&\equiv \varrho_{3,2,1}(\mathcal{E}_i),\ \text{representation } (\varrho_{3,2,1},\mathscr{W}_{3,2,1})\equiv(\rho_3\otimes\rho_2\otimes\rho_1,\mathscr{V}_3\otimes\mathscr{V}_2\otimes\mathscr{V}_1)\end{aligned}$$

$$\begin{aligned}(\rho_4\otimes\rho_3\otimes\rho_2\otimes\rho_1)(\mathcal{E}_i) &= (\rho_4\otimes\varrho_{3,2,1})(\mathcal{E}_i)\\
&= \rho_4(\mathcal{E}_i)\otimes I_{\mathscr{W}_{3,2,1}}+I_{\mathscr{V}_4}\otimes\varrho_{3,2,1}(\mathcal{E}_i)\\
&= \rho_4(\mathcal{E}_i)\otimes I_{\mathscr{V}_3\otimes\mathscr{V}_2\otimes\mathscr{V}_1}+I_{\mathscr{V}_4}\otimes(\rho_3(\mathcal{E}_i)\otimes I_{\mathscr{V}_2}\otimes I_{\mathscr{V}_1}\\
&\quad +I_{\mathscr{V}_3}\otimes\rho_2(\mathcal{E}_i)\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_3}\otimes I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i))\\
&= \rho_4(\mathcal{E}_i)\otimes I_{\mathscr{V}_3}\otimes I_{\mathscr{V}_2}\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_4}\otimes\rho_3(\mathcal{E}_i)\otimes I_{\mathscr{V}_2}\otimes I_{\mathscr{V}_1}\\
&\quad +I_{\mathscr{V}_4}\otimes I_{\mathscr{V}_3}\otimes\rho_2(\mathcal{E}_i)\otimes I_{\mathscr{V}_1}+I_{\mathscr{V}_4}\otimes I_{\mathscr{V}_3}\otimes I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i)\end{aligned}$$

$$\begin{aligned}(\rho_L\otimes\rho_{L-1}\otimes\ldots\otimes\rho_1)(\mathcal{E}_i) &= \rho_L(\mathcal{E}_i)\otimes I_{\mathscr{V}_{L-1}}\otimes I_{\mathscr{V}_{L-2}}\otimes\ldots\otimes I_{\mathscr{V}_1}\\
&\quad +I_{\mathscr{V}_L}\otimes\rho_{L-1}(\mathcal{E}_i)\otimes I_{\mathscr{V}_{L-2}}\otimes I_{\mathscr{V}_{L-3}}\otimes\ldots\otimes I_{\mathscr{V}_1}\\
&\quad +I_{\mathscr{V}_L}\otimes I_{\mathscr{V}_{L-1}}\otimes\rho_{L-2}(\mathcal{E}_i)\otimes I_{\mathscr{V}_{L-3}}\otimes I_{\mathscr{V}_{L-4}}\otimes\ldots\otimes I_{\mathscr{V}_1}\\
&\quad +\ldots\\
&\quad +I_{\mathscr{V}_L}\otimes I_{\mathscr{V}_{L-1}}\otimes\ldots\otimes I_{\mathscr{V}_2}\otimes\rho_1(\mathcal{E}_i)\\
&= \sum_{n=1}^{L} I_{\mathscr{V}_L}\otimes I_{\mathscr{V}_{L-1}}\otimes\ldots\otimes I_{\mathscr{V}_{n+1}}\otimes\rho_n(\mathcal{E}_i)\otimes I_{\mathscr{V}_{n-1}}\otimes I_{\mathscr{V}_{n-2}}\otimes\ldots\otimes I_{\mathscr{V}_1}\end{aligned}$$

Problem 2

(a)

$$\begin{aligned}\mathbb{S}^\alpha &= \sum_{i=1}^{L} S_i^\alpha \\ &= S_1^\alpha + S_2^\alpha\end{aligned}$$

$$\begin{aligned}\implies \hat{C} &= \sum_{\alpha=1}^{3} \left(S_1^\alpha + S_2^\alpha\right)\left(S_1^\alpha + S_2^\alpha\right) \\ &= \sum_{\alpha=1}^{3} \left(S_1^\alpha S_1^\alpha + S_1^\alpha S_2^\alpha + S_2^\alpha S_1^\alpha + S_2^\alpha S_2^\alpha\right) \\ &= \sum_{\alpha=1}^{3} \left(\left(S^\alpha \otimes I_2\right)\left(S^\alpha \otimes I_2\right) + \left(S^\alpha \otimes I_2\right)\left(I_2 \otimes S^\alpha\right) + \left(I_2 \otimes S^\alpha\right)\left(S^\alpha \otimes I_2\right) + \left(I_2 \otimes S^\alpha\right)\left(I_2 \otimes S^\alpha\right)\right) \\ &= \sum_{\alpha=1}^{3} \left(\left(S^\alpha\right)^2 \otimes I_2 + S^\alpha \otimes S^\alpha + S^\alpha \otimes S^\alpha + I_2 \otimes \left(S^\alpha\right)^2\right)\end{aligned}$$

$$\left(S^\alpha\right)^2 = \left(\frac{\hbar \sigma^\alpha}{2}\right)^2 = \frac{\hbar^2}{4} I_2$$

$$\begin{aligned}\implies \hat{C} &= \sum_{\alpha=1}^{3} \left(\frac{\hbar^2}{4} I_2 \otimes I_2 + 2S^\alpha \otimes S^\alpha + I_2 \otimes \frac{\hbar^2}{4} I_2\right) \\ &= \frac{3\hbar^2}{2} I_4 + 2\sum_{\alpha=1}^{3} S^\alpha \otimes S^\alpha\end{aligned}$$

$$\begin{aligned}S^1 \otimes S^1 &= \frac{\hbar\sigma^1}{2} \otimes \frac{\hbar\sigma^1}{2} \\ &= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\end{aligned}
\qquad
\begin{aligned}S^2 \otimes S^2 &= \frac{\hbar\sigma^2}{2} \otimes \frac{\hbar\sigma^2}{2} \\ &= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\end{aligned}
\qquad
\begin{aligned}S^3 \otimes S^3 &= \frac{\hbar\sigma^3}{2} \otimes \frac{\hbar\sigma^3}{2} \\ &= \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\implies \hat{C} &= \frac{\hbar^2}{2}\left(3I_4 + \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3\right) \\ &= \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \\ &= 2\hbar^2 \left|1\right\rangle\left\langle 1\right| + \hbar^2 \left|2\right\rangle\left\langle 2\right| + \hbar^2 \left|3\right\rangle\left\langle 3\right| + 2\hbar^2 \left|4\right\rangle\left\langle 4\right| + \hbar^2 \left|2\right\rangle\left\langle 3\right| + \hbar^2 \left|3\right\rangle\left\langle 2\right|\end{aligned}$$

$$\begin{aligned}\mathbb{S}^3 &= S_1^3 + S_2^3 \\ &= \frac{\hbar}{2}\sigma^3 \otimes I_2 + I_2 \otimes \frac{\hbar}{2}\sigma^3 \\ &= \frac{\hbar}{2}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right) \\ &= \hbar\left(\left|1\right\rangle\left\langle 1\right| - \left|4\right\rangle\left\langle 4\right|\right)\end{aligned}$$

We can see how $\mathbb{S}^3$ acts on the basis vectors

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |1\rangle \qquad |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |2\rangle$$

$$|\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \qquad |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |4\rangle$$

and construct our eigenvectors as a linear combination of these by inspection.

$$\begin{aligned} \mathbb{S}^3 |\uparrow\uparrow\rangle &= \hbar \left(|1\rangle \langle 1| - |4\rangle \langle 4|\right) |1\rangle & \mathbb{S}^3 |\uparrow\downarrow\rangle &= \hbar \left(|1\rangle \langle 1| - |4\rangle \langle 4|\right) |2\rangle \\ &= \hbar \left(|1\rangle - 0\right) & &= \hbar \left(0 - 0\right) \\ &= \hbar |\uparrow\uparrow\rangle & &= 0 \\ \mathbb{S}^3 |\downarrow\uparrow\rangle &= \hbar \left(|1\rangle \langle 1| - |4\rangle \langle 4|\right) |3\rangle & \mathbb{S}^3 |\downarrow\downarrow\rangle &= \hbar \left(|1\rangle \langle 1| - |4\rangle \langle 4|\right) |4\rangle \\ &= \hbar \left(0 - 0\right) & &= \hbar \left(0 - |4\rangle\right) \\ &= 0 & &= -\hbar |\downarrow\downarrow\rangle \end{aligned}$$

Thus the basis vectors are all eigenvectors of $\mathbb{S}^3$. Since $\hat{C}$ and $\mathbb{S}^3$ commute, they must share eigenvectors. We can see how $\hat{C}$ acts on the eigenvectors of $\mathbb{S}^3$ and construct a linear combination of them if necessary to find the eigenvectors of $\hat{C}$.

$$\begin{aligned} \hat{C} |\uparrow\uparrow\rangle &= \left(2\hbar^2 |1\rangle \langle 1| + \hbar^2 |2\rangle \langle 2| + \hbar^2 |3\rangle \langle 3| + 2\hbar^2 |4\rangle \langle 4| + \hbar^2 |2\rangle \langle 3| + \hbar^2 |3\rangle \langle 2|\right) |1\rangle \\ &= 2\hbar^2 |1\rangle + 0 + 0 + 0 + 0 + 0 \\ &= 2\hbar^2 |\uparrow\uparrow\rangle \\ \hat{C} |\uparrow\downarrow\rangle &= \left(2\hbar^2 |1\rangle \langle 1| + \hbar^2 |2\rangle \langle 2| + \hbar^2 |3\rangle \langle 3| + 2\hbar^2 |4\rangle \langle 4| + \hbar^2 |2\rangle \langle 3| + \hbar^2 |3\rangle \langle 2|\right) |2\rangle \\ &= 0 + \hbar^2 |2\rangle + 0 + 0 + 0 + \hbar^2 |3\rangle \\ &= \hbar^2 \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ \hat{C} |\downarrow\uparrow\rangle &= \left(2\hbar^2 |1\rangle \langle 1| + \hbar^2 |2\rangle \langle 2| + \hbar^2 |3\rangle \langle 3| + 2\hbar^2 |4\rangle \langle 4| + \hbar^2 |2\rangle \langle 3| + \hbar^2 |3\rangle \langle 2|\right) |3\rangle \\ &= 0 + 0 + \hbar^2 |3\rangle + \hbar^2 |2\rangle \\ &= \hbar^2 \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ \hat{C} |\downarrow\downarrow\rangle &= \left(2\hbar^2 |1\rangle \langle 1| + \hbar^2 |2\rangle \langle 2| + \hbar^2 |3\rangle \langle 3| + 2\hbar^2 |4\rangle \langle 4| + \hbar^2 |2\rangle \langle 3| + \hbar^2 |3\rangle \langle 2|\right) |4\rangle \\ &= 2\hbar^2 |4\rangle \\ &= 2\hbar^2 |\downarrow\downarrow\rangle \end{aligned}$$

Thus $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are eigenvectors each with corresponding eigenvalue $2\hbar$. We can make a linear combination of the other two basis vectors to find the remaining eigenvectors.

$$\begin{aligned} \hat{C} \left(a |\uparrow\downarrow\rangle + b |\downarrow\uparrow\rangle\right) &= \hbar^2 \left((a+b) |\uparrow\downarrow\rangle + (a+b) |\uparrow\downarrow\rangle\right) \\ a = b \implies a\, \hat{C} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) &= 2a\, \hbar^2 \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ a = -b \implies a\, \hat{C} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) &= 0 \end{aligned}$$

This shows that $|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$ are eigenvectors of $\hat{C}$, with eigenvalues $2\hbar^2$ and 0. Thus all the (normalised) eigenvectors and corresponding eigenvalues are

$$\vec{v}_1 = |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_1 = 2\hbar^2 \qquad \vec{v}_2 = |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda_2 = 2\hbar^2$$

$$\vec{v}_3 = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_3 = 2\hbar^2 \qquad \vec{v}_4 = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda_4 = 0$$

(b)

$$\begin{aligned}
H &= J \sum_{i=1}^{L} \sum_{\alpha=1}^{3} S_i^\alpha\, S_{i+1}^\alpha \\
&= J \sum_{\alpha=1}^{3} \left(S_1^\alpha\, S_2^\alpha + S_2^\alpha\, S_1^\alpha\right) \\
&= J \sum_{\alpha=1}^{3} \left(S_1^\alpha\, S_1^\alpha + S_1^\alpha\, S_2^\alpha + S_2^\alpha\, S_1^\alpha + S_2^\alpha\, S_2^\alpha - S_1^\alpha\, S_1^\alpha - S_2^\alpha\, S_2^\alpha\right) \\
&= J \left(\hat{C} - \frac{3\hbar^2}{2}\, I_4\right)
\end{aligned}$$

$$\begin{aligned}
H\,\vec{v}_a &= J\left(\hat{C}\,\vec{v}_a - \frac{3\hbar^2}{2}\, I_4\,\vec{v}_a\right) \\
&= J\left(\lambda_a\,\vec{v}_a - \frac{3\hbar^2}{2}\,\vec{v}_a\right) \\
&= J\left(\lambda_a - \frac{3\hbar^2}{2}\right)\vec{v}_a \\
&= \lambda'_a\,\vec{v}_a,
\end{aligned}$$

i.e. H and $\hat{C}$ have the same eigenvectors. We can thus easily find the eigenvalues λ' of H.

$$\begin{aligned}
\lambda_{1,2,3} &= 2\hbar^2 \\
\implies \lambda'_{1,2,3} &= \frac{J\,\hbar^2}{2}
\end{aligned} \qquad \begin{aligned}
\lambda_4 &= 0 \\
\implies \lambda'_4 &= -\frac{3J\,\hbar^2}{2}
\end{aligned}$$

Thus the spectrum of H is $\left\{\frac{J\,\hbar^2}{2}, -\frac{3J\,\hbar^2}{2}\right\}$.

Problem 3

(i)

$$
\begin{aligned}
\hat{X}(s) &= \hat{V}(s)\,\hat{X}\,\hat{V}^{\dagger}(s) \\
\implies \frac{d\hat{X}(s)}{ds} &= \frac{d\hat{V}(s)}{ds}\,\hat{X}\,\hat{V}^{\dagger}(s) + \hat{V}(s)\,\hat{X}\,\frac{d\hat{V}^{\dagger}(s)}{ds} \\
&= i\,\hat{P}\,\hat{V}(s)\,\hat{X}\,\hat{V}^{\dagger}(s) - i\,\hat{V}(s)\,\hat{X}\,\hat{P}\,\hat{V}^{\dagger}(s) \\
&= -i\left(\hat{V}(s)\,\hat{X}\,\hat{P}\,\hat{V}^{\dagger}(s) - \hat{V}(s)\,\hat{P}\,\hat{X}\,\hat{V}^{\dagger}(s)\right) && \text{as } \hat{P} \text{ and } \hat{V}(s) \text{ commute} \\
&= -i\,\hat{V}(s)\left[\hat{X},\hat{P}\right]\hat{V}^{\dagger}(s) \\
&= -i\,\hat{V}(s)\cdot\left(i\,\hbar\,\hat{I}\right)\cdot\hat{V}^{\dagger}(s) \\
&= \hbar\,\hat{I} \\
\implies \hat{X}(s) &= \int \hbar\,\hat{I}\,ds \\
&= s\,\hbar\,\hat{I} + \hat{C} \\
\hat{X}(0) &= \hat{V}(0)\,\hat{X}\,\hat{V}^{\dagger}(0) \\
&= \hat{X} \\
\text{Also, } \hat{X}(0) &= \hat{C} \\
\implies \hat{X}(s) &= s\,\hbar\,\hat{I} + \hat{X}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\text{Label } \hat{F}(r,s,t) &\equiv \hat{U}(rt)\,\hat{V}(rt) \\
&= e^{i\,r\,t\,\hat{X}}\,e^{i\,r\,s\,\hat{P}} \\
\frac{\partial \hat{F}(r,s,t)}{\partial r} &= i\,t\,\hat{X}\,e^{i\,r\,t\,\hat{X}}\,e^{i\,r\,s\,\hat{P}} + e^{i\,r\,t\,\hat{X}}\left(i\,s\,\hat{P}\right)e^{i\,r\,s\,\hat{P}} \\
&= i\,t\,\hat{X}\,\hat{F}(r,s,t) + i\,s\,e^{i\,r\,t\,\hat{X}}\,\hat{P}\,e^{-i\,r\,t\,\hat{X}}\,e^{i\,r\,t\,\hat{X}}\,e^{i\,r\,s\,\hat{P}} \\
&= i\,t\,\hat{X}\,\hat{F}(r,s,t) + i\,s\,\hat{U}(rt)\,\hat{P}\,\hat{U}^{\dagger}(rt)\,\hat{F}(r,s,t)
\end{aligned}
$$

$$
\begin{aligned}
\text{Label } \hat{P}(rt) &\equiv \hat{U}(rt)\,\hat{P}\,\hat{U}^\dagger(rt) \\
\frac{\partial \hat{P}(rt)}{\partial t} &= \frac{\partial \hat{U}(rt)}{\partial t}\,\hat{P}\,\hat{U}^\dagger(rt) + \hat{U}(rt)\,\hat{P}\,\frac{\partial \hat{U}^\dagger(rt)}{\partial t} \\
&= i\,r\,\hat{X}\,\hat{U}(rt)\,\hat{P}\,\hat{U}^\dagger(rt) - i\,r\,\hat{U}(rt)\,\hat{P}\,\hat{X}\,\hat{U}^\dagger(rt) \\
&= i\,r\left(\hat{U}(rt)\,\hat{X}\,\hat{P}\,\hat{U}^\dagger(rt) - \hat{U}(rt)\,\hat{P}\,\hat{X}\,\hat{U}^\dagger(rt)\right) \\
&= i\,r\,\hat{U}(rt)\left[\hat{X},\hat{P}\right]\hat{U}^\dagger(rt) \\
&= -r\,\hbar\,\hat{I} \\
\implies \hat{P}(rt) &= -\int r\,\hbar\,\hat{I}\,dt \\
&= -r\,t\,\hbar\,\hat{I} + \hat{C}_1 \\
\hat{P}(0) &= \hat{U}(0)\,\hat{P}\,\hat{U}^\dagger(0) \\
&= \hat{P} \\
\text{Also, } \hat{P}(0) &= \hat{C}_1 \\
\implies \hat{P}(rt) &= -r\,t\,\hbar\,\hat{I} + \hat{P}
\end{aligned}
$$

$$
\begin{aligned}
\implies \frac{\partial \hat{F}(r,s,t)}{\partial r} &= \left(i\,t\,\hat{X} + i\,s\,\hat{P} - i\,r\,s\,t\,\hbar\,\hat{I}\right)\hat{F}(r,s,t) \\
\implies \hat{F}(r,s,t) &= \hat{C}_2\,e^{i\,r\,t\,\hat{X} + i\,r\,s\,\hat{P} - \frac{i\,r^2\,s\,t\,\hbar}{2}\,\hat{I} + \hat{C}_3} \\
\hat{F}(r,s,t) &= \text{product of two unitary matrices} \\
\implies \hat{C}_2 &= \hat{I} \\
\hat{F}(0,s,t) &= \hat{U}(0)\,\hat{V}(0) \\
&= \hat{I} \\
\text{Also, } \hat{F}(0,s,t) &= e^{\hat{C}_3} \\
\implies \hat{C}_3 &= 0 \\
\implies \hat{F}(r,s,t) &= e^{i\,r\,t\,\hat{X} + i\,r\,s\,\hat{P} - \frac{i\,r^2\,s\,t\,\hbar}{2}\,\hat{I}} \\
\hat{U}(t)\,\hat{V}(s) &= \hat{F}(1,s,t) \\
&= e^{i\,t\,\hat{X} + i\,s\,\hat{P} - \frac{i\,s\,t\,\hbar}{2}\,\hat{I}}
\end{aligned}
$$

(iii)

We can find $\hat{V}(s)\,\hat{U}(t)$ by using a similar method to that in (ii). If we label $\hat{G}(r,s,t) \equiv \hat{V}(rs)\,\hat{U}(rt)$ and calculate $\frac{\partial \hat{G}(r,s,t)}{\partial r}$, we result in an expression in terms of $\hat{X}(rs)$. Substituting our answer from (i) and proceeding in a similar fashion in (ii) gives us

$$\hat{V}(s)\,\hat{U}(t) = e^{i\,t\,\hat{X} + i\,s\,\hat{P} + \frac{i\,s\,t\,\hbar}{2}\,\hat{I}}.$$

We know that $\left[\hat{A}, \alpha\,\hat{I}\right] = 0$ for any operator $\hat{A}$ and constant α. We also know that, for any commuting matrices A and B, we have $e^A\,e^B = e^{A+B}$. Thus, by inspection,

$$
\begin{aligned}
e^{-i\,s\,t\,\hbar\,\hat{I}}\,e^{i\,t\,\hat{X} + i\,s\,\hat{P} + \frac{i\,s\,t\,\hbar}{2}\,\hat{I}} &= e^{i\,t\,\hat{X} + i\,s\,\hat{P} - \frac{i\,s\,t\,\hbar}{2}\,\hat{I}} \\
\implies e^{-i\,s\,t\,\hbar\,\hat{I}}\,\hat{V}(s)\,\hat{U}(t) &= \hat{U}(t)\,\hat{V}(s) \\
\implies \hat{W} &= e^{-i\,s\,t\,\hbar\,\hat{I}}
\end{aligned}
$$