MAU34403: Quantum Mechanics I
Homework 2 due 30/09/2021

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Problem 1

$$
\begin{aligned}
\hat{a}^\dagger |n\rangle &= \hat{U} a^\dagger \hat{U}^\dagger |n\rangle \\
&= \sum_{i,j} U_{ij} |i\rangle \langle j| a^\dagger \sum_{k,l} U^*_{lk} |k\rangle \langle l|n\rangle \\
&= \sum_{i,j,k} U_{ij} U^*_{nk} |i\rangle \langle j| a^\dagger |k\rangle \\
&= \sum_{i,j,k} \sqrt{k+1}\, U_{ij} U^*_{nk} |i\rangle \langle j|k+1\rangle \\
&= \sum_{i,k} \sqrt{k+1}\, U_{i,k+1} U^*_{nk} |i\rangle \\
\text{also } \hat{a}^\dagger |n\rangle &= \sqrt{n+1}\, e^{-i\phi_n} |n+1\rangle \\
\implies U_{i,k+1} U^*_{nk} &= \begin{cases} e^{-i\phi_n} \text{ if } i = n+1, k = n \\ 0\, \forall\, i \neq n+1, k \neq n \end{cases}
\end{aligned}
$$

Since this holds for any n, the only entries in $\hat{U}$ that are non-zero are the diagonal entries, i.e. $\hat{U}$ is diagonal. Since $\hat{U}$ is unitary, we must have that

$$
\begin{aligned}
\hat{U}\,\hat{U}^\dagger &= \hat{I} \\
\implies U_{ii} U^*_{ii} &= 1 \\
\implies |U_{ii}| &= 1.
\end{aligned}
$$

We can thus express any entry of the diagonal matrix as $U_{jj} = e^{i\theta_j}$. Since we also have that $U_{j+1,j+1} U^*_{jj} = e^{-i\phi_j}$. We can then do the calculation

$$
\begin{aligned}
U_{j+1,j+1} U^*_{jj} &= e^{-i\phi_j} \\
&= e^{i(\theta_{j+1} - \theta_j)} \\
\implies \theta_j - \theta_{j+1} &= \phi_j.
\end{aligned}
$$

Since we are asked to find any unitary operator that relates the two sets of creation and annihilation operators, we can define one of the θ_j to be an arbitrary real number. Setting

$$
\begin{aligned}
&\theta_0 = 0, \\
&\theta_1 = \theta_0 - \phi_0 = -\phi_0, \\
&\theta_2 = \theta_1 - \phi_1 = -\phi_0 - \phi_1, \\
&\theta_3 = \theta_2 - \phi_2 = -\phi_0 - \phi_1 - \phi_2, \\
&\dots \\
\implies &\theta_j = -\sum_{l=1}^{j} \phi_{l-1}.
\end{aligned}
$$

This gives us a unitary operator

$$
\hat{U} = \sum_{j,k=0}^{\infty} \delta_{jk} e^{i\theta_j} |j\rangle \langle k| \qquad \text{where } \theta_j = -\sum_{l=1}^{j} \phi_{l-1}, \text{ with } \theta_0 \equiv 0
$$

Problem 2

$$\begin{aligned}
\hat{L}_1 \left|n\right\rangle &= \frac{i}{2}\left(\hat{a}\hat{a}\left|n\right\rangle + \hat{a}^\dagger\hat{a}^\dagger\left|n\right\rangle\right) \\
&= \frac{i}{2}\left(\sqrt{n}\, e^{i\phi_{n-1}}\hat{a}\left|n-1\right\rangle + \sqrt{n+1}\, e^{-i\phi_n}\hat{a}^\dagger\left|n+1\right\rangle\right) \\
&= \frac{i}{2}\left(\sqrt{n(n-1)}\, e^{i(\phi_{n-1}+\phi_{n-2})}\left|n-2\right\rangle + \sqrt{(n+1)(n+2)}\, e^{-i(\phi_n+\phi_{n+1})}\left|n+2\right\rangle\right)
\end{aligned}$$

$$\begin{aligned}
\hat{L}_2 \left|n\right\rangle &= \frac{i}{2}\left(\hat{a}^\dagger\hat{a}\left|n\right\rangle + \hat{a}\hat{a}^\dagger\left|n\right\rangle\right) \\
&= \frac{i}{2}\left(\sqrt{n}\, e^{i\phi_{n-1}}\hat{a}^\dagger\left|n-1\right\rangle + \sqrt{n+1}\, e^{-i\phi_n}\hat{a}^\dagger\left|n+1\right\rangle\right) \\
&= \frac{i}{2}\left(\sqrt{n}\sqrt{n}\, e^{i(\phi_{n-1}-\phi_{n-1})}\left|n\right\rangle + \sqrt{n+1}\sqrt{n+1}\, e^{-i(\phi_n-\phi_n)}\left|n\right\rangle\right) \\
&= \frac{i(2n+1)}{2}\left|n\right\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{L}_3 \left|n\right\rangle &= \frac{1}{2}\left(\hat{a}\hat{a}\left|n\right\rangle - \hat{a}^\dagger\hat{a}^\dagger\left|n\right\rangle\right) \\
&= \frac{1}{2}\left(\sqrt{n}\, e^{i\phi_{n-1}}\hat{a}\left|n-1\right\rangle - \sqrt{n+1}\, e^{-i\phi_n}\hat{a}^\dagger\left|n+1\right\rangle\right) \\
&= \frac{1}{2}\left(\sqrt{n(n-1)}\, e^{i(\phi_{n-1}+\phi_{n-2})}\left|n-2\right\rangle - \sqrt{(n+1)(n+2)}\, e^{-i(\phi_n+\phi_{n+1})}\left|n+2\right\rangle\right)
\end{aligned}$$

Problem 3

$$\begin{aligned}
\hat{L}_1^2 &= -\frac{1}{4}\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right)\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\hat{a}\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\hat{a}^\dagger+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)+\hat{a}^\dagger\hat{a}+\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\hat{I}+\hat{a}^\dagger\hat{a}+\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\hat{a}+3\hat{a}^\dagger\hat{a}+2\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= -\frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}+2\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+4\hat{a}^\dagger\hat{a}+2\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)
\end{aligned}$$

$$\begin{aligned}
\hat{L}_2^2 &= -\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\hat{I}\right)\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\hat{I}\right)\\
&= -\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\frac{1}{2}\hat{a}^\dagger\hat{a}+\frac{1}{2}\hat{a}^\dagger\hat{a}+\frac{1}{4}\hat{I}\right)\\
&= -\left(\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\hat{I}\right)\\
&= -\left(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+2\hat{a}^\dagger\hat{a}+\frac{1}{4}\hat{I}\right)
\end{aligned}$$

$$\begin{aligned}
\hat{L}_3^2 &= \frac{1}{4}\left(\hat{a}\hat{a}-\hat{a}^\dagger\hat{a}^\dagger\right)\left(\hat{a}\hat{a}-\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\hat{a}\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)-\hat{a}^\dagger\hat{a}-\hat{I}-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}-\hat{a}^\dagger\hat{a}-\hat{a}^\dagger\hat{a}-\hat{I}-\hat{a}^\dagger\hat{a}-\hat{I}-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a}+\hat{I}\right)\hat{a}-3\hat{a}^\dagger\hat{a}-2\hat{I}-\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)\\
&= \frac{1}{4}\left(\hat{a}\hat{a}\hat{a}\hat{a}-2\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}-4\hat{a}^\dagger\hat{a}-2\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\right)
\end{aligned}$$

$$\begin{aligned}
\hat{C} &= \hat{L}_1^2-\hat{L}_2^2+\hat{L}_3^2\\
&= -\frac{1}{4}\hat{a}\hat{a}\hat{a}\hat{a}-\frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}-\hat{a}^\dagger\hat{a}-\frac{1}{2}\hat{I}-\frac{1}{4}\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\\
&\quad +2\hat{a}^\dagger\hat{a}+\frac{1}{4}\hat{I}+\frac{1}{4}\hat{a}\hat{a}\hat{a}\hat{a}-\frac{1}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}-\hat{a}^\dagger\hat{a}-\frac{1}{2}\hat{I}+\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\\
&= -\frac{3}{4}\hat{I}
\end{aligned}$$

Problem 4

$$\begin{aligned}
\hat{b}_1(t) &= \hat{U}_1(t)\,\hat{a}\,\hat{U}_1^\dagger(t) \\
\frac{d\hat{b}_1}{dt} &= \left(\frac{d\hat{U}_1}{dt}\right)\hat{a}\,\hat{U}_1^\dagger(t) + \hat{U}_1(t)\,\hat{a}\left(\frac{d\hat{U}_1^\dagger}{dt}\right)
\end{aligned}$$

$$\begin{aligned}
\hat{U}_1(t) &= e^{t\hat{L}_1} \\
&= \sum_{k=0}^{\infty}\frac{t^k}{k!}\hat{L}_1^j \\
&= 1 + \sum_{k=1}^{\infty}\frac{t^k}{k!}\hat{L}_1^k \\
\frac{d\hat{U}_1}{dt} &= 0 + \sum_{k=1}^{\infty}\frac{k\,t^{k-1}}{k!}\hat{L}_1^k \\
&= \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}\hat{L}_1^k \\
&= \left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\hat{L}_1^k\right)\hat{L}_1 \\
&= \hat{U}_1(t)\,\hat{L}_1
\end{aligned}$$

$$\begin{aligned}
\hat{U}_1^\dagger(t) &= e^{t\hat{L}_1^\dagger} \\
&= e^{-t\hat{L}_1} \\
&= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k\,t^k}{k!}\hat{L}_1^k \\
\frac{d\hat{U}_1^\dagger}{dt} &= 0 + \sum_{k=1}^{\infty}\frac{k(-1)^k\,t^{k-1}}{k!}\hat{L}_1^k \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k\,t^{k-1}}{(k-1)!}\hat{L}_1^k \\
&= -\hat{L}_1\left(\sum_{k=0}^{\infty}\frac{(-1)^k\,t^k}{k!}\hat{L}_1^k\right) \\
&= -\hat{L}_1\,\hat{U}_1^\dagger(t)
\end{aligned}$$

$$\begin{aligned}
\implies \frac{d\hat{b}_1}{dt} &= \hat{U}_1(t)\,\hat{L}_1\,\hat{a}\,\hat{U}_1^\dagger(t) - \hat{U}_1(t)\,\hat{a}\,\hat{L}_1\,\hat{U}_1^\dagger(t) \\
&= \hat{U}_1(t)\left[\hat{L}_1,\hat{a}\right]\hat{U}_1^\dagger(t)
\end{aligned}$$

Similarly, since $\hat{b}_1^\dagger(t) = \hat{U}_1(t)\,\hat{a}^\dagger\,\hat{U}_1^\dagger(t)$, and the above manipulations do not depend on or alter $\hat{a}$,

$$\frac{d\hat{b}_1^\dagger}{dt} = \hat{U}_1(t)\left[\hat{L}_1,\hat{a}^\dagger\right]\hat{U}_1^\dagger(t)$$

$$\begin{aligned}
\left[\hat{L}_1,\hat{a}\right] &= \left[\frac{i}{2}\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right),\hat{a}\right] \\
&= \frac{i}{2}\left([\hat{a}\hat{a},\hat{a}] + \left[\hat{a}^\dagger\hat{a}^\dagger,\hat{a}\right]\right) \\
&= \frac{i}{2}\left(0 + \hat{a}^\dagger\left[\hat{a}^\dagger,\hat{a}\right] + \left[\hat{a}^\dagger,\hat{a}\right]\hat{a}^\dagger\right) \\
&= \frac{i}{2}\left(-\hat{a}^\dagger\,\hat{I} - \hat{I}\,\hat{a}^\dagger\right) \\
&= -i\,\hat{a}^\dagger
\end{aligned}$$

$$\begin{aligned}
\left[\hat{L}_1,\hat{a}^\dagger\right] &= \left[\frac{i}{2}\left(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger\right),\hat{a}^\dagger\right] \\
&= \frac{i}{2}\left(\left[\hat{a}\hat{a},\hat{a}^\dagger\right] + \left[\hat{a}^\dagger\hat{a}^\dagger,\hat{a}^\dagger\right]\right) \\
&= \frac{i}{2}\left(\hat{a}\left[\hat{a},\hat{a}^\dagger\right] + \left[\hat{a},\hat{a}^\dagger\right]\hat{a} + 0\right) \\
&= \frac{i}{2}\left(\hat{a}\,\hat{I} + \hat{I}\,\hat{a}\right) \\
&= i\,\hat{a}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{b}_1}{dt} &= \hat{U}_1(t)\left(-i\,\hat{a}^\dagger\right)\hat{U}_1^\dagger(t) \\
&= -i\,\hat{b}_1^\dagger(t) \\
\frac{d^2\hat{b}_1}{dt^2} &= -i\,\frac{d\hat{b}_1^\dagger}{dt} \\
&= -i\left(i\,\hat{b}_1(t)\right) \\
&= \hat{b}_1(t)
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{b}_1^\dagger}{dt} &= \hat{U}_1(t)\left(i\,\hat{a}\right)\hat{U}_1^\dagger(t) \\
&= i\,\hat{b}_1(t) \\
\frac{d^2\hat{b}_1^\dagger}{dt^2} &= i\,\frac{d\hat{b}_1}{dt} \\
&= i\left(-i\,\hat{b}_1^\dagger(t)\right) \\
&= \hat{b}_1^\dagger(t)
\end{aligned}$$

$$
\begin{aligned}
\text{Let } \hat{b}_1(t) &= c_1(t)\hat{a} + d_1(t)\hat{a}^\dagger \\
\frac{d\hat{b}_1}{dt} &= \frac{dc_1}{dt}\,\hat{a} + \frac{dd_1}{dt}\,\hat{a}^\dagger \\
\text{Also, } \frac{d\hat{b}_1}{dt} &= -i\,\hat{b}_1^\dagger(t) \\
&= -i\,c_1^*(t)\,\hat{a}^\dagger - i\,d_1^*(t)\,\hat{a} \\
\implies \frac{dc_1}{dt} &= -i\,d_1^*(t) \\
\text{and } \frac{dd_1}{dt} &= -i\,c_1^*(t)
\end{aligned}
$$

Assume an ansatz for $c_1(t)$ and $d_1(t)$ of the form

$$
c_1(t) = \alpha_c\,e^{\chi_c t} + \beta_c\,e^{-\psi_c t}, \qquad d_1(t) = \alpha_d\,e^{\chi_d t} + \beta_d\,e^{-\psi_d t}.
$$

Since we know

$$
\frac{d^2c_1}{dt^2} = c_1(t), \qquad \frac{d^2d_1}{dt^2} = d_1(t),
$$

we know that $\chi_c = \psi_c = \chi_d = \psi_d = 1$, i.e. t does not have a coefficient. This gives us

$$
c_1(t) = \alpha_c\,e^{t} + \beta_c\,e^{-t}, \qquad d_1(t) = \alpha_d\,e^{t} + \beta_d\,e^{-t}.
$$

Since we know

$$
\begin{aligned}
\frac{dc_1}{dt} &= -i\,d_1^* \\
\implies \alpha_c\,e^t - \beta_c\,e^t &= -i\,\alpha_d^*\,e^{-t} - i\,\beta_d^*\,e^t \\
\implies \alpha_c &= -i\,\beta_d^* \\
\text{and } \beta_c &= i\,\alpha_d^*.
\end{aligned}
$$

This gives us

$$
c_1(t) = -i\,\beta_d^*\,e^t + i\,\alpha_d^*\,e^{-t} \qquad d_1(t) = \alpha_d\,e^t + \beta_d\,e^{-t}.
$$

We can equate the two expressions we have for $\hat{b}_1^\dagger(t)$ and pick some value for t to solve for the remaining unknown parameters, i.e.

$$
\begin{aligned}
\hat{b}_1^\dagger(t) &= \hat{U}_1(t)\,\hat{a}^\dagger\,\hat{U}_1^\dagger(t) \\
&= e^{t\hat{L}_1}\,\hat{a}^\dagger\,e^{-t\hat{L}_1} \\
\text{Also, } \hat{b}_1^\dagger(t) &= c_1^*(t)\,\hat{a}^\dagger + d_1^*(t)\,\hat{a} \\
\hat{b}_1^\dagger(0) = e^0\,\hat{a}^\dagger\,e^0 &= c_1^*(0)\,\hat{a}^\dagger + d_1^*(0)\,\hat{a} \\
\implies d_1^*(0) = \alpha_d^* + \beta_d^* &= 0 \\
\implies \beta_d &= -\alpha_d \\
\text{Also, } c_1^*(0) = i\,\beta_d - i\,\alpha_d &= 1 \\
\implies \alpha_d &= \frac{i}{2} \\
\implies \beta_d &= -\frac{i}{2}
\end{aligned}
$$

We can then rewrite $c_1(t)$ and $d_1(t)$ as

$$
\begin{aligned}
c_1(t) &= -i\,\frac{i}{2}\,e^t + i\left(-\frac{i}{2}\right)e^{-t} & d_1(t) &= \frac{i}{2}\,e^t - \frac{i}{2}\,e^{-t} \\
&= \frac{1}{2}\left(e^t + e^{-t}\right) & &= \frac{i}{2}\left(e^t - e^{-t}\right) \\
&= \cosh(t) & &= i\sinh(t)
\end{aligned}
$$

We can therefore express $\hat{b}_1(t)$ and $\hat{b}_1^\dagger(t)$ as

$$\hat{b}_1(t) = \cosh(t)\,\hat{a} + i\sinh(t)\,\hat{a}^\dagger \qquad\qquad \hat{b}_1^\dagger(t), = \cosh(t)\,\hat{a}^\dagger - i\sinh(t)\,\hat{a}.$$

$$\begin{aligned}
\left[\hat{b}_1(t), \hat{b}_1^\dagger(t)\right] &= \left[\cosh(t)\,\hat{a} + i\sinh(t)\,\hat{a}^\dagger, \cosh(t)\,\hat{a}^\dagger - i\sinh(t)\,\hat{a}\right] \\
&= \left[\cosh(t)\,\hat{a}, \cosh(t)\,\hat{a}^\dagger\right] + \left[\cosh(t)\,\hat{a}, -i\sinh(t)\,\hat{a}\right] \\
&\quad + \left[i\sinh(t)\,\hat{a}^\dagger, \cosh(t)\,\hat{a}^\dagger\right] + \left[i\sinh(t)\hat{a}^\dagger, -i\sinh(t)\,\hat{a}\right] \\
&= \cosh^2(t)\left[\hat{a}, \hat{a}^\dagger\right] - i\cosh(t)\sinh(t)\left[\hat{a}, \hat{a}\right] + i\cosh(t)\sinh(t)\left[\hat{a}^\dagger, \hat{a}^\dagger\right] + \sinh^2(t)\left[\hat{a}^\dagger, \hat{a}\right] \\
&= \cosh^2(t)\,\hat{I} - 0 + 0 - \sinh^2(t)\,\hat{I} \\
&= \hat{I},
\end{aligned}$$

and so $\hat{b}_1(t)$ and $\hat{b}_1^\dagger(t)$ satisfy the canonical commutation relation.