MAU34403: Quantum Mechanics I
Homework 1 due 21/09/2021

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JS Theoretical Physics

Problem 1

From inspecting the basis matrices E_{ab} and investigating the multiplication of these basis matrices $E_{ab}E_{cd}$, one can see that the multiplication results in a non-zero matrix only if both the kth column of E_{ab} and kth row of E_{cd} contain a 1; otherwise the multiplication results in a zero matrix. One can thus deduce that

$$\begin{aligned}
E_{ab}E_{cd} &= \delta_{bc}E_{ad} && \text{by investigation}\\
&= \sum_{ij} f^{ij}_{ab,cd}E_{ij} && \text{by definition}\\
\implies f^{ij}_{ab,cd} &= \delta_{ai}\delta_{dj}\delta_{bc}
\end{aligned}$$

Problem 2

- Closure: $\det(AB) = \det(A)\det(B) = 1 \implies AB \in SU(n)$
- Associativity: $A(BC) = (AB)C$ under standard matrix multiplication
- Identity: $\det(I_n) = 1 \implies I_n \in SU(n)$, $AI_n = I_nA = A \implies I_n =$ identity element
- Inverse: since all matrices in $SU(n)$ are unitary, then for any $A \in SU(n)$ there must exist a corresponding inverse, where $A^{-1} = A^\dagger$. We have that $\det\left(A^{-1}\right) = \frac{1}{\det A} = 1$, and so $A^{-1} \in SU(n)$.

Problem 3

- To check if the matrix commutator is bilinear

$$\begin{aligned}
[\mathcal{S}+\mathcal{T},\mathcal{U}] &= (\mathcal{S}+\mathcal{T})\,\mathcal{U} - \mathcal{U}\,(\mathcal{S}+\mathcal{T})\\
&= \mathcal{S}\,\mathcal{U} + \mathcal{T}\,\mathcal{U} - \mathcal{U}\,\mathcal{S} - \mathcal{U}\,\mathcal{T}\\
&= \mathcal{S}\,\mathcal{U} - \mathcal{U}\,\mathcal{S} + \mathcal{T}\,\mathcal{U} - \mathcal{U}\,\mathcal{T}\\
&= [\mathcal{S},\mathcal{U}] + [\mathcal{T},\mathcal{U}] \implies \text{right distributivity}\\
[\mathcal{U},\mathcal{S}+\mathcal{T}] &= -[\mathcal{S}+\mathcal{T},\mathcal{U}]\\
&= -\left([\mathcal{S},\mathcal{U}] + [\mathcal{T},\mathcal{U}]\right)\\
&= [\mathcal{U},\mathcal{S}] + [\mathcal{U},\mathcal{T}] \implies \text{left distributivity}\\
[a\,\mathcal{S}, b\,\mathcal{T}] &= (a\,\mathcal{S})(b\,\mathcal{T}) - (b\,\mathcal{T})(a\,\mathcal{S})\\
&= a\,b\,(\mathcal{S}\,\mathcal{T} - \mathcal{T}\,\mathcal{S})\\
&= a\,b[\mathcal{S},\mathcal{T}] \implies \text{compatability with scalars}
\end{aligned}$$

- To check if closure exists

$$\begin{aligned}
([\mathcal{A},\mathcal{B}])^\dagger &= (\mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A})^\dagger\\
&= \mathcal{B}^\dagger\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{B}^\dagger\\
&= (-\mathcal{B})(-\mathcal{A}) - (-\mathcal{A})(-\mathcal{B})\\
&= \mathcal{B}\,\mathcal{A} - \mathcal{A}\,\mathcal{B}\\
&= -[\mathcal{A},\mathcal{B}]\\
\mathrm{tr}([\mathcal{A},\mathcal{B}]) &= \mathrm{tr}(\mathcal{A}\,\mathcal{B} - \mathcal{B}\,\mathcal{A})\\
&= \mathrm{tr}(\mathcal{A}\,\mathcal{B}) - \mathrm{tr}(\mathcal{B}\,\mathcal{A})\\
&= 0\\
\mathcal{A},\mathcal{B} \in \mathfrak{su}(n) &\implies [\mathcal{A},\mathcal{B}] \in \mathfrak{su}(n)
\end{aligned}$$

- To check if the matrix commutator is skew symmetric

$$[\mathcal{J},\mathcal{J}] = \mathcal{J}\mathcal{J} - \mathcal{J}\mathcal{J} = 0$$

- To check if the matrix commutator satisfies the Jacobi identity

$$\begin{aligned}
[\mathcal{J},[\mathcal{K},\mathcal{L}]] + [\mathcal{K}[\mathcal{L},\mathcal{J}]] + [\mathcal{L},[\mathcal{J},\mathcal{K}]] &= [\mathcal{J},\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K}] + [\mathcal{K},\mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L}] + [\mathcal{L},\mathcal{J}\mathcal{K} - \mathcal{K}\mathcal{J}] \\
&= \mathcal{J}(\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K}) - (\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K})\mathcal{J} + \mathcal{K}(\mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L}) \\
&\quad - (\mathcal{L}\mathcal{J} - \mathcal{J}\mathcal{L})\mathcal{K} + \mathcal{L}(\mathcal{J}\mathcal{K} - \mathcal{K}\mathcal{J}) - (\mathcal{J}\mathcal{K} - \mathcal{K}\mathcal{J})\mathcal{L} \\
&= \mathcal{J}\mathcal{K}\mathcal{L} - \mathcal{J}\mathcal{L}\mathcal{K} - \mathcal{K}\mathcal{L}\mathcal{J} + \mathcal{L}\mathcal{K}\mathcal{J} + \mathcal{K}\mathcal{L}\mathcal{J} - \mathcal{K}\mathcal{J}\mathcal{L} \\
&\quad - \mathcal{L}\mathcal{J}\mathcal{K} + \mathcal{J}\mathcal{L}\mathcal{K} + \mathcal{L}\mathcal{J}\mathcal{K} - \mathcal{L}\mathcal{K}\mathcal{J},\mathcal{J} - \mathcal{J}\mathcal{K}\mathcal{L} + \mathcal{K}\mathcal{J}\mathcal{L} \\
&= \mathcal{O}
\end{aligned}$$

The dimension of $\mathfrak{su}(n)$ is equal to the number of basis matrices that spans the Lie algebra, and so we can find the dimension by counting the number of basis matrices.

Let's first consider the matrices that contain off-diagonal entries only. For an $n \times n$ matrix, there are $n^2 - n$ off-diagonal entries. Since the Lie algebra is over $\mathbb{R}$, then each entry a_{jk} has two independent parameters r_{jk} and c_{jk}, where $a_{jk} = r_{jk} + c_{jk}i$. However, since the matrices in this Lie algebra are anti-hermitian, we have the relation that $a_{jk} = -a_{kj}^*$, i.e. $r_{jk} = -r_{kj}$ and $c_{jk} = c_{kj}$. Thus half of the $n^2 - n$ off-diagonal entries each have 2 independent parameters, and so the off-diagonals contribute $n^2 - n$ to the dimension.

Now let's consider the matrices that contain diagonal entries only. For an $n \times n$ matrix, there are n diagonal entries. Since the matrix is anti-hermitian, we have that $r_{jj} = -r_{jj}$ and $c_{jj} = c_{jj}$. Thus, the diagonal may only contain imaginary entries, and so for each entry there is a maximum of one independent parameter. Given $n-1$ of these entries, there is only one possible value that the remaining entry can take for the matrix to be traceless (the negative of the sum of these entries), and thus one of the diagonal entries is not independent. Thus the diagonal entries contribute $n-1$ to the dimension.

$$\begin{aligned}
\dim \mathfrak{su}(n) &= n^2 - n + n - 1 \\
&= n^2 - 1
\end{aligned}$$

Problem 4

A.

(i)

$$\begin{aligned}
W &= \sum_{i,j=1}^{n} W_{ij} \left|i\right\rangle \left\langle j\right| \\
&= \sum_{i,j=1}^{n} \left\langle i\right| W \left|j\right\rangle \left|i\right\rangle \left\langle j\right| \\
&= \sum_{i,j=1}^{n} \left\langle i|j+1\right\rangle \left|i\right\rangle \left\langle j\right| \\
&= \sum_{i,j=1}^{n} \delta_{j+1}^{i} \left|i\right\rangle \left\langle j\right| \text{ where } W_{i,n+1} = W_{i1} \\
&= \left|1\right\rangle \left\langle n\right| + \sum_{i=1}^{n-1} \left|i+1\right\rangle \left\langle i\right| \\
n = 4 \implies W &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

(ii)

$$\begin{aligned}
W^\dagger W &= \left(|1\rangle\langle n| + \sum_{i=1}^{n-1} |i+1\rangle\langle i|\right)^\dagger \left(|1\rangle\langle n| + \sum_{j=1}^{n-1} |j+1\rangle\langle j|\right) \\
&= \left(|n\rangle\langle 1| + \sum_{i=1}^{n-1} |i\rangle\langle i+1|\right) \left(|1\rangle\langle n| + \sum_{j=1}^{n-1} |j+1\rangle\langle j|\right) \\
&= |n\rangle\langle 1|1\rangle\langle n| + \sum_{j=1}^{n-1} |n\rangle\langle 1|j+1\rangle\langle j| + \sum_{i=1}^{n-1} |i\rangle\langle i+1|1\rangle\langle n| + \sum_{i,j=1}^{n-1} |i\rangle\langle i+1|j+1\rangle\langle j| \\
&= |n\rangle\langle n| + 0 + 0 + \sum_{i=1}^{n-1} |i\rangle\langle i| = \sum_{i=1}^{n} |i\rangle\langle i| = \mathcal{I} \implies W \text{ is unitary}
\end{aligned}$$

B.

(i)

$$\begin{aligned}
V \text{ unitary} &\implies VV^\dagger = V^\dagger V = \mathcal{I} \\
V \text{ diagonal} &\implies V^t = V \\
&\implies V^\dagger = V^* \\
&\implies VV^* = \mathcal{I} \\
&\implies 1 \cdot 1^* = qq^* = q^2\left(q^2\right)^* = \ldots = q^{n-1}\left(q^{n-1}\right)^* = 1 \\
(q^m)^* = (q^*)^m &\implies qq^* = (qq^*)^2 = \ldots = (qq^*)^{n-1} = 1 \\
&\implies |q| = 1
\end{aligned}$$

(ii)

$$\begin{aligned}
V \text{ traceless} &\implies 1 + q + q^2 + \ldots + q^{n+1} = 0 \\
&\implies \frac{1-q^n}{1-q} = 0 \\
&\implies q^n = 1,\, q \neq 1
\end{aligned}$$

(iii)

If V is traceless then we have that $q^n = 1$ and $q \neq 1$. This satisfies the condition for V to be unitary that $|q| = 1$ as long as $n \neq 1$, and so V can indeed be unitary and traceless.

(iv)

$V_{11} = q^0$, $V_{22} = q^1$, $\ldots$, $V_{nn} = q^{n-1}$, $V_{ij} = 0\,\forall\, i \neq j \implies V_{ij} = \delta_{ij} q^{i-1}$

$$V = \sum_{i,j=1}^{n} \delta_{ij} q^{i-1} |i\rangle\langle j|$$

C.

$$
\begin{aligned}
VW &= \sum_{i=1}^{n} q^{i-1} \left|i\right\rangle \left\langle i\right| \left(\sum_{j=1}^{n-1} \left|j+1\right\rangle \left\langle j\right| + \left|1\right\rangle \left\langle n\right| \right) \\
&= \sum_{i=1}^{n} q^{i-1} \left|i\right\rangle \left\langle i\right| \sum_{j=1}^{n-1} \left|j+1\right\rangle \left\langle j\right| + \sum_{i=1}^{n} q^{i-1} \left|i\right\rangle \left\langle i|1\right\rangle \left\langle n\right| \\
&= \left(\sum_{i=1}^{n-1} q^{i-1} \left|i\right\rangle \left\langle i\right| + q^{n-1} \left|n\right\rangle \left\langle n\right| \right) \sum_{j=1}^{n-1} \left|j+1\right\rangle \left\langle j\right| + \left|1\right\rangle \left\langle n\right| \\
&= \sum_{i,j=1}^{n-1} q^{i-1} \left|i\right\rangle \left\langle i|j+1\right\rangle \left\langle j\right| + \sum_{j=1}^{n-1} q^{n-1} \left|n\right\rangle \left\langle n|j+1\right\rangle \left\langle j\right| + \left|1\right\rangle \left\langle n\right| \\
&= \sum_{i=2}^{n-1} q^{i-1} \left|i\right\rangle \left\langle i-1\right| + q^{n-1} \left|n\right\rangle \left\langle n-1\right| + \left|1\right\rangle \left\langle n\right| \\
&= \left|1\right\rangle \left\langle n\right| + \sum_{i=2}^{n} q^{i-1} \left|i\right\rangle \left\langle i-1\right| \\
&= \left|1\right\rangle \left\langle n\right| + \sum_{i=1}^{n-1} q^{i} \left|i+1\right\rangle \left\langle i\right|
\end{aligned}
$$

$$
\begin{aligned}
WV &= \left(\sum_{i=1}^{n-1} \left|i+1\right\rangle \left\langle i\right| + \left|1\right\rangle \left\langle n\right| \right) \sum_{j=1}^{n} q^{j-1} \left|j\right\rangle \left\langle j\right| \\
&= \sum_{i=1}^{n-1} \left|i+1\right\rangle \left\langle i\right| \sum_{j=1}^{n} q^{j-1} \left|j\right\rangle \left\langle j\right| + \sum_{j=1}^{n} q^{j-1} \left|1\right\rangle \left\langle n|j\right\rangle \left\langle j\right| \\
&= \sum_{i=1}^{n-1} \left|i+1\right\rangle \left\langle i\right| \left(\sum_{j=1}^{n-1} q^{j-1} \left|j\right\rangle \left\langle j\right| + q^{n-1} \left|n\right\rangle \left\langle n\right| \right) + q^{n-1} \left|1\right\rangle \left\langle n\right| \\
&= \sum_{i,j=1}^{n-1} q^{j-1} \left|i+1\right\rangle \left\langle i|j\right\rangle \left\langle j\right| + \sum_{i=1}^{n-1} q^{n-1} \left|i+1\right\rangle \left\langle i|n\right\rangle \left\langle n\right| + q^{n-1} \left|1\right\rangle \left\langle n\right| \\
&= q^{n-1} \left|1\right\rangle \left\langle n\right| + \sum_{i=1}^{n-1} q^{i-1} \left|i+1\right\rangle \left\langle i\right|
\end{aligned}
$$

D.

$$
VW = cWV \implies \left|1\right\rangle \left\langle n\right| + \sum_{i=1}^{n-1} q^{i} \left|i+1\right\rangle \left\langle i\right| = c \left(q^{n-1} \left|1\right\rangle \left\langle n\right| + \sum_{i=1}^{n-1} q^{i-1} \left|i+1\right\rangle \left\langle i\right| \right)
$$

$$
1 = c\,q^{n-1} \qquad q = c \qquad q^2 = c\,q \qquad \ldots \qquad q^{n-1} = c\,q^{n-2}
$$

$$
c\,q^{n-1} = 1 \qquad\qquad c = q
$$

Thus if we choose a q such that $q^n = 1$ and let $c = q$ then we have that $VW = WV$.