MAU34404: Quantum Mechanics II
Homework 6 due 08/04/2022

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JS Theoretical Physics

Problem 1

1.

$$\begin{aligned}
\rho^\dagger &= \left(\sum_n p_n \left|\psi_n\right\rangle \left\langle\psi_n\right|\right)^\dagger \\
&= \sum_n p_n \left(\left|\psi_n\right\rangle \left\langle\psi_n\right|\right)^\dagger \\
&= \sum_n p_n \left|\psi_n\right\rangle \left\langle\psi_n\right| \\
&= \rho
\end{aligned}$$

2.

$$\begin{aligned}
\langle\phi|\rho|\phi\rangle &= \sum_n p_n \langle\phi|\psi_n\rangle \langle\psi_n|\phi\rangle \\
&= \sum_n p_n \langle\phi|\psi_n\rangle \langle\phi|\psi_n\rangle^* \\
&= \sum_n p_n \left|\langle\phi|\psi_n\rangle\right|^2 \\
&\geqslant 0 \text{ as } p_n \geqslant 0 \text{ and } \left|\langle\phi|\psi_n\rangle\right|^2 \geqslant 0
\end{aligned}$$

3.

$$\begin{aligned}
\rho &= \rho^2 \\
\sum_n p_n \left|\psi_n\right\rangle \left\langle\psi_n\right| &= \left(\sum_l p_l \left|\psi_l\right\rangle \left\langle\psi_l\right|\right)\left(\sum_m p_m \left|\psi_m\right\rangle \left\langle\psi_m\right|\right) \\
&= \sum_{l,m} p_l p_m \langle\psi_l|\psi_m\rangle \left|\psi_l\right\rangle \left\langle\psi_m\right| \\
&= \sum_n p_n^2 \left|\psi_n\right\rangle \left\langle\psi_n\right| + \sum_{l\neq m} p_l p_m \langle\psi_l|\psi_m\rangle \left|\psi_l\right\rangle \left\langle\psi_m\right| \qquad (\langle\psi_n|\psi_n\rangle = 1)
\end{aligned}$$

Since the second term on the RHS does not take the form of any term on the LHS, it thus must be 0. We then have that $p_i = p_i^2$ for all i. This is only true if p_i is either 0 or 1. However, the sum of all p_i is equal to 1, so we can deduce that there is some j such that $p_j = 1$, and $p_i = 0 \iff i \neq j$. Thus the system is in state $\left|\psi_j\right\rangle$, and therefore a pure state.

Problem 2

1.

$$\begin{aligned}
\rho &= \sum p_n \left|\psi_n\right\rangle \left\langle\psi_n\right| \\
&= \frac{1}{3}\left[\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| + \left(\frac{4}{5}\left|0\right\rangle + \frac{3}{5}\left|1\right\rangle\right)\left(\frac{4}{5}\left\langle 0\right| + \frac{3}{5}\left\langle 1\right|\right)\right] \\
\rho &= \frac{41}{75}\left|0\right\rangle\left\langle 0\right| + \frac{34}{75}\left|1\right\rangle\left\langle 1\right| + \frac{4}{25}\left(\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|\right)
\end{aligned}$$

2.

$$\begin{aligned}
\langle H\rangle &= \sum_n p_n \left\langle\psi_n|H|\psi_n\right\rangle \\
&= \frac{1}{3}\left[\langle 0|H|0\rangle + \langle 1|H|1\rangle + \left(\frac{4}{5}\left\langle 0\right| + \frac{3}{5}\left\langle 1\right|\right) H \left(\frac{4}{5}\left|0\right\rangle + \frac{3}{5}\left|1\right\rangle\right)\right] \\
&= \frac{1}{3}\left\{\left\langle 0\right|\hbar\omega\left(0+\frac{1}{2}\right)\left|0\right\rangle + \left\langle 1\right|\hbar\omega\left(1+\frac{1}{2}\right)\left|1\right\rangle + \left(\frac{4}{5}\left\langle 0\right| + \frac{3}{5}\left\langle 1\right|\right)\left[\frac{4}{5}\hbar\omega\left(0+\frac{1}{2}\right)\left|0\right\rangle + \frac{3}{5}\hbar\omega\left(1+\frac{1}{2}\right)\right]\right\} \\
&= \frac{\hbar\omega}{3}\left(\frac{41}{50}\langle 0|0\rangle + \frac{51}{25}\langle 1|1\rangle + \frac{18}{25}\langle 0|1\rangle + \frac{6}{25}\langle 1|0\rangle\right) \\
\langle H\rangle &= \frac{143}{150}\hbar\omega
\end{aligned}$$

3.

$$\begin{aligned}
0 &= \begin{vmatrix} \frac{41}{75}-\lambda & \frac{4}{25} \\ \frac{4}{25} & \frac{34}{75}-\lambda \end{vmatrix} \\
&= \left(\frac{41}{75}-\lambda\right)\left(\frac{34}{75}-\lambda\right) - \frac{4}{25}\frac{4}{25} \\
&= \lambda^2 - \lambda + \frac{2}{9} \\
\implies \lambda &= \frac{1}{3}, \frac{2}{3} \\
S &= -k_B \sum_i \lambda_i \ln \lambda_i \\
&= -k_B\left(\frac{1}{3}\ln\frac{1}{3} + \frac{2}{3}\ln\frac{2}{3}\right) \\
S &= k_B\left(\ln 3 - \frac{2}{3}\ln 2\right)
\end{aligned}$$

4.

$$\begin{aligned}
\rho' &= \sum_\alpha \Lambda_\alpha \rho \Lambda_\alpha \\
&= \left|0\right\rangle\left\langle 0\right|\rho\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\rho\left|1\right\rangle\left\langle 1\right| \\
&= \left|0\right\rangle\left\langle 0\right|\left(\frac{41}{75}\left|0\right\rangle + \frac{4}{25}\left|1\right\rangle\right)\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\left(\frac{34}{75}\left|1\right\rangle + \frac{4}{25}\left|0\right\rangle\right)\left\langle 1\right| \\
\rho' &= \frac{41}{75}\left|0\right\rangle + \frac{34}{75}\left|1\right\rangle
\end{aligned}$$