MAU34404: Quantum Mechanics II
Homework 5 due 01/04/2022

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JS Theoretical Physics

Problem 1

1.

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\begin{aligned}
L &= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \\
&= \frac{1}{2}m\left(\dot{q}_c + \dot{y}\right)^2 - \frac{1}{2}m\omega^2 (q_c + y)^2 \\
&= \frac{1}{2}m\dot{q}_c^2 + m\dot{q}_c\dot{y} + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 q_c^2 - m\omega^2 q_c y - \frac{1}{2}m\omega^2 y^2 \\
&= \left(\frac{1}{2}m\dot{q}_c^2 - \frac{1}{2}m\omega^2 q_c^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) + m\left(\dot{q}_c\dot{y} + \ddot{q}_c y\right) && (-\omega^2 q_c = \ddot{q}_c) \\
&= \left(\frac{1}{2}m\dot{q}_c^2 - \frac{1}{2}m\omega^2 q_c^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) + m\frac{d}{d\tau}(\dot{q}_c y)
\end{aligned}
$$

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\begin{aligned}
\int_t^{t'} L(q,\dot{q})\,d\tau &= \int_t^{t'} \left(\frac{1}{2}m\dot{q}_c^2 - \frac{1}{2}m\omega^2 q_c^2\right) d\tau + \int_t^{t'} \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) d\tau + \int_t^{t'} m\frac{d}{d\tau}(\dot{q}_c y)\,d\tau \\
&= I[q_c] + \frac{m}{2}\int_t^{t'} \left(\dot{y}^2 - \omega^2 y^2\right) d\tau + m\dot{q}_c y\Big|_t^{t'} \\
&= I[q_c] + \frac{m}{2}\int_t^{t'} \left(\dot{y}^2 - \omega^2 y^2\right) d\tau && (y(t') = y(t) = 0)
\end{aligned}
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2.

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\begin{aligned}
\dot{y}^2 &= \left(\sum_{n=1}^{K} \dot{y}_n\right)^2 \\
&= \left(\sum_{n=1}^{K} a_n \frac{n\pi}{T}\cos\left[\frac{n\pi(\tau - t)}{T}\right]\right)^2 \\
&= \sum_{m,n=1}^{K} a_m a_n mn\frac{\pi^2}{T^2}\cos\left[\frac{m\pi(\tau - t)}{T}\right]\cos\left[\frac{n\pi(\tau - t)}{T}\right] \\
&= \sum_{n=1}^{K} a_n^2 \left(\frac{n\pi}{T}\right)^2 \cos^2\left[\frac{n\pi(\tau - t)}{T}\right] + \sum_{\substack{m,n=1 \\ m\neq n}}^{K} a_m a_n \frac{mn\pi^2}{T^2}\cos\left[\frac{m\pi(\tau - t)}{T}\right]\cos\left[\frac{n\pi(\tau - t)}{T}\right]
\end{aligned}
$$

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\begin{aligned}
\omega^2 y^2 &= \omega^2 \left(\sum_{n=1}^{K} y_n \right)^2 \\
&= \omega^2 \left(\sum_{n=1}^{K} a_n \sin\left[\frac{n\pi(\tau - t)}{T} \right] \right)^2 \\
&= \omega^2 \sum_{m,n=1}^{K} a_m a_n \sin\left[\frac{m\pi(\tau - t)}{T} \right] \sin\left[\frac{n\pi(\tau - t)}{T} \right] \\
&= \sum_{n=1}^{K} a_n^2 \omega^2 \sin^2\left[\frac{n\pi(\tau - t)}{T} \right] + \sum_{\substack{m,n=1 \\ m\neq n}}^{K} a_m a_n \omega^2 \sin\left[\frac{m\pi(\tau - t)}{T} \right] \sin\left[\frac{n\pi(\tau - t)}{T} \right]
\end{aligned}
$$

$$
\begin{aligned}
\int_t^{t'} \left(\dot{y}^2 - \omega^2 y^2 \right) d\tau = & \int_t^{t'} \sum_{n=1}^{K} a_n^2 \left\{ \left(\frac{n\pi}{T} \right)^2 \cos^2\left[\frac{n\pi(\tau - t)}{T} \right] - \omega^2 \sin^2\left[\frac{n\pi(\tau - t)}{T} \right] \right\} d\tau \\
& + \int_t^{t'} \sum_{\substack{m,n=1 \\ m\neq n}}^{K} a_m a_n \frac{mn\pi^2}{T^2} \cos\left[\frac{m\pi(\tau - t)}{T} \right] \cos\left[\frac{n\pi(\tau - t)}{T} \right] d\tau \\
& - \int_t^{t'} \sum_{\substack{m,n=1 \\ m\neq n}}^{K} a_m a_n \omega^2 \sin\left[\frac{m\pi(\tau - t)}{T} \right] \sin\left[\frac{n\pi(\tau - t)}{T} \right] d\tau \\
= & \sum_{n=1}^{K} a_n^2 \int_0^T \left[\left(\frac{n\pi}{T} \right)^2 \cos^2\left(\frac{n\pi\tau}{T} \right) - \omega^2 \sin^2\left(\frac{n\pi\tau}{T} \right) \right] d\tau \\
& + \sum_{\substack{m,n=1 \\ m\neq 1}}^{K} a_m a_n \frac{mn\pi^2}{T^2} \int_0^T \cos\left(\frac{m\pi\tau}{T} \right) \cos\left(\frac{n\pi\tau}{T} \right) d\tau \\
& - \sum_{\substack{m,n=1 \\ m\neq 1}}^{K} a_m a_n \omega^2 \int_0^T \sin\left(\frac{m\pi\tau}{T} \right) \sin\left(\frac{n\pi\tau}{T} \right) d\tau \qquad (T \equiv t' - t) \\
= & \sum_{n=1}^{K} a_n^2 \frac{T}{n\pi} \int_0^{n\pi} \left[\left(\frac{n\pi}{T} \right)^2 \cos^2 u - \omega^2 \sin^2 u \right] du \\
& + \sum_{\substack{m,n=1 \\ m\neq 1}}^{K} a_m a_n \frac{mn\pi^2}{T^2} \frac{T}{\pi} \int_0^{\pi} \cos(mv) \cos(nv)\, dv \\
& - \sum_{\substack{m,n=1 \\ m\neq 1}}^{K} a_m a_n \omega^2 \frac{T}{\pi} \int_0^{\pi} \sin(mv) \sin(nv)\, dv \qquad (u \equiv \tfrac{n\pi\tau}{T},\ v \equiv \tfrac{\pi\tau}{T}) \\
= & \; T \sum_{n=1}^{K} \frac{a_n^2}{n\pi} \frac{1}{2} \left\{ \left(\frac{n\pi}{T} \right)^2 \left[u + \frac{\sin(2u)}{2} \right] - \omega^2 \left[u - \frac{\sin(2u)}{2} \right] \right\}_{u=0}^{u=n\pi} + 0 - 0 \\
& \qquad \text{(orthogonality of sin and cos)} \\
= & \frac{T}{2} \sum_{n=1}^{K} \frac{a_n^2}{n\pi} \left\{ \left(\frac{n\pi}{T} \right)^2 \left[n\pi + \frac{\sin(2n\pi)}{2} \right] - \omega^2 \left[n\pi - \frac{\sin(2n\pi)}{2} \right] \right\} \\
= & \frac{T}{2} \sum_{n=1}^{K} a_n^2 \left[\left(\frac{n\pi}{T} \right)^2 - \omega^2 \right] \qquad (\sin(2n\pi) = 0)
\end{aligned}
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