MAU34404: Quantum Mechanics II
Homework 4 due 04/02/2022

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

1.

$$\begin{aligned}
\frac{\partial \Phi_n}{\partial t} &= \frac{1}{2}\sqrt{\frac{w}{2}}\left(-\frac{2\,v}{w^2}\right)\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \sqrt{\frac{2}{w}}\cos\!\left(\frac{n\,\pi\,x}{w}\right)\left(-\frac{n\,\pi\,x\,v}{w^2}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right]\left[-\frac{i\,v}{2\,\hbar\,w^2}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right) + \frac{i}{2\,\hbar\,w}\left(-2\,E_n^i\,a\right)\right] \\
&= \left[-\frac{v}{w^2}\sqrt{\frac{w}{2}}\sqrt{\frac{w}{2}} - \frac{n\,\pi\,v\,x}{w^2}\frac{\cos\left(\frac{n\,\pi\,x}{w}\right)}{\sin\left(\frac{n\,\pi\,x}{w}\right)} - \frac{i\,v}{2\,\hbar\,w^2}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right) + \frac{i}{2\,\hbar\,w}\left(-2\,E_n^i\,a\right)\right] \\
&\quad \times \sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&= \left[-\frac{v}{2\,w} - \frac{n\,\pi\,v\,x}{w^2}\cot\!\left(\frac{n\,\pi\,x}{w}\right) - \frac{i\,m\,v^2\,x^2}{2\,\hbar\,w^2} + \frac{i\,n^2\,\pi^2\,\hbar\,v\,t}{2\,m\,a\,w^2} - \frac{i\,n^2\,\pi^2\,\hbar}{2\,m\,a\,w}\right]\Phi_n
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Phi_n}{\partial x} &= \frac{n\,\pi}{w}\sqrt{\frac{2}{w}}\cos\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \frac{i\,m\,v\,x}{\hbar\,w}\sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
\frac{\partial^2 \Phi_n}{\partial x^2} &= -\frac{n\,\pi}{w}\sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\frac{n\,\pi}{w}\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \frac{n\,\pi}{w}\sqrt{\frac{2}{w}}\cos\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right]\frac{i\,m\,v\,x}{\hbar\,w} \\
&\quad + \frac{i\,m\,v}{\hbar\,w}\sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \frac{i\,m\,v\,x}{\hbar\,w}\sqrt{\frac{2}{w}}\cos\!\left(\frac{n\,\pi\,x}{w}\right)\frac{n\,\pi}{w}\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&\quad + \frac{i\,m\,v\,x}{\hbar\,w}\sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right]\frac{i\,m\,v\,x}{\hbar\,w} \\
&= \left[-\frac{n^2\,\pi^2}{w^2} + \frac{i\,m\,n\,\pi\,v\,x}{\hbar\,w^2}\frac{\cos\left(\frac{n\,\pi\,x}{w}\right)}{\sin\left(\frac{n\,\pi\,x}{w}\right)} + \frac{i\,m\,v}{\hbar\,w} + \frac{i\,m\,n\,\pi\,v\,x}{\hbar\,w^2}\frac{\cos\left(\frac{n\,\pi\,x}{w}\right)}{\sin\left(\frac{n\,\pi\,x}{w}\right)} - \frac{m^2\,v^2\,x^2}{\hbar^2\,w^2}\right] \\
&\quad \times \sqrt{\frac{2}{w}}\sin\!\left(\frac{n\,\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - 2\,E_n^i\,a\,t\right)\right] \\
&= \left[-\frac{n^2\,\pi^2}{w^2} + \frac{2\,i\,m\,n\,\pi\,v\,x}{\hbar\,w^2}\cot\!\left(\frac{n\,\pi\,x}{w}\right) + \frac{i\,m\,v}{\hbar\,w} - \frac{m^2\,v^2\,x^2}{\hbar^2\,w^2}\right]\Phi_n
\end{aligned}$$

$$
\begin{aligned}
i\,\hbar\,\frac{\partial\Phi_n}{\partial t} + \frac{\hbar^2}{2\,m}\,\frac{\partial^2\Phi_n}{\partial x^2} &= \Bigg[-\frac{i\,\hbar\,v}{2\,w} - \frac{i\,n\,\pi\,\hbar\,v\,x}{w^2}\cot\Big(\frac{n\,\pi\,x}{w}\Big) + \frac{m\,v^2\,x^2}{2\,w^2} - \frac{n^2\,\pi^2\,\hbar^2\,v\,t}{2\,m\,a\,w^2} + \frac{n^2\,\pi^2\,\hbar^2}{2\,m\,a\,w} \\
&\quad - \frac{n^2\,\pi^2\,\hbar^2}{2\,m\,w^2} + \frac{i\,n\,\pi\,\hbar\,v\,x}{w^2}\cot\Big(\frac{n\,\pi\,x}{w}\Big) + \frac{i\,\hbar\,v}{2\,w} - \frac{m\,v^2\,x^2}{2\,w^2}\Bigg]\,\Phi_n \\
&= \left[\frac{n^2\,\pi^2\,\hbar^2}{2\,m\,a\,w^2}\,(-v\,t + w - a)\right]\Phi_n \\
&= 0 \\
\implies i\,\hbar\,\frac{\partial\Phi_n}{\partial t} &= -\,\frac{\hbar^2}{2m}\,\frac{\partial^2\Phi_n}{\partial x^2}
\end{aligned}
$$

2.

$$
\begin{aligned}
\Psi(x,0) &= \sum_{n=1}^{\infty} c_n\,\Phi_n(x,0) \\
\sqrt{\frac{2}{a}}\sin\Big(\frac{\pi\,x}{a}\Big) &= \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{a}}\sin\Big(\frac{n\,\pi\,x}{a}\Big)\exp\left(\frac{i\,m\,v\,x^2}{2\,\hbar\,a}\right) \\
\frac{2}{a}\sin\Big(\frac{\pi\,x}{a}\Big)\sin\left(\frac{n'\,\pi\,x}{a}\right)\exp\left(-\frac{i\,m\,v\,x^2}{2\,\hbar\,a}\right) &= \sum_{n=1}^{\infty} c_n\,\frac{2}{a}\sin\Big(\frac{n\,\pi\,x}{a}\Big)\sin\left(\frac{n'\,\pi\,x}{a}\right) \\
\frac{2}{a}\int_0^a \exp\left(-\frac{i\,m\,v\,x^2}{2\,\hbar\,a}\right)\sin\left(\frac{n'\,\pi\,x}{a}\right)\sin\Big(\frac{\pi\,x}{a}\Big)\,dx &= \sum_{n=1}^{\infty} c_n\,\frac{2}{a}\int_0^a \sin\Big(\frac{n\,\pi\,x}{a}\Big)\sin\left(\frac{n'\,\pi\,x}{a}\right)dx \\
\frac{2}{a}\int_0^\pi \exp\left(-\frac{i\,m\,v}{2\,\hbar\,a}\,\frac{u^2\,a^2}{\pi^2}\right)\sin\left(\frac{n'\,\pi}{a}\,\frac{u\,a}{\pi}\right)\sin\Big(\frac{\pi}{a}\,\frac{u\,a}{\pi}\Big)\;du\,\frac{a}{\pi} &= \sum_{n=1}^{\infty} c_n\,\frac{2}{a}\,\frac{a}{2}\,\delta_n^{n'} \\
\frac{2}{\pi}\int_0^\pi \exp\left(-i\,\frac{m\,v\,a}{2\,\pi^2\,\hbar}\,u^2\right)\sin(n'u)\sin(u)\,du &= c_{n'} \\
\implies c_n &= \frac{2}{\pi}\int_0^\pi e^{-i\,a\,z^2}\;\sin(n\,z)\sin(z)\,dz
\end{aligned}
$$

3.

$$
\begin{aligned}
2\,a &= w(T) \\
&= a + v\,T \\
\implies T &= \frac{a}{v}
\end{aligned}
$$

$$
\begin{aligned}
C\exp(\pm i\,\omega\,t) &= \exp\left[\frac{i}{2\,\hbar\,a}\left(m\,v\,x^2 - \frac{2\,\pi^2\,\hbar^2\,a\,t}{2\,m\,a^2}\right)\right] \\
&= \exp\left(\frac{i\,m\,v\,x^2}{2\,\hbar\,a}\right)\exp\left(-i\,\frac{\pi^2\,\hbar}{2\,m\,a^2}\,t\right) \\
\implies \omega &= \frac{\pi^2\,\hbar}{2\,m\,a^2} \\
\tau &= \frac{2\,\pi}{\omega} \\
&= \frac{4\,m\,a^2}{\pi\,\hbar}
\end{aligned}
$$

$$
\begin{aligned}
& T \gg \tau \\
\implies & \frac{a}{v} \gg \frac{4\,m\,a^2}{\pi\,\hbar} \\
\implies & \alpha = \frac{m\,v\,a}{2\,\pi^2\,\hbar} \ll \frac{1}{8\,\pi} < 1 \\
\implies & \alpha \ll 1 \implies e^{-i\,\alpha\,z^2} \approx 1
\end{aligned}
$$

4.

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} e^{-i\,\alpha\,z^2}\;\sin(n\,z)\sin(z)\,dz \\ &\approx \frac{2}{\pi}\int_0^{\pi} \sin(n\,z)\sin(z)\,dz \\ &= \frac{2}{\pi}\,\frac{\pi}{2}\,\delta_n^1 \\ &= \delta_n^1 \end{aligned}$$

$$\begin{aligned} \Psi(x,t) &= \sum_{n=1}^{\infty} \delta_n^1\,\Phi_n(x,t) \\ &= \Phi_1(x,t) \\ &= \sqrt{\frac{2}{w}}\,\sin\!\left(\frac{\pi\,x}{w}\right)\exp\!\left[\frac{i}{2\,\hbar\,w}\left(m\,v\,x^2 - \frac{\pi^2\,\hbar\,t}{m\,a}\right)\right] \end{aligned}$$

$$\begin{aligned} \frac{a}{v} \gg \frac{4\,m\,a^2}{\pi\,\hbar} \quad &\Longrightarrow \quad \frac{m\,v\,a}{2\,\hbar} \ll \frac{\pi}{8} < 1 \\ \frac{m\,v\,x^2}{2\,\hbar\,w} \leqslant \frac{m\,v\,a^2}{2\,\hbar\,a} &= \frac{m\,v\,a}{2\,\hbar} \ll 1 \end{aligned}$$

$$\Longrightarrow\quad \Psi(x,t) \approx \sqrt{\frac{2}{w}}\,\sin\!\left(\frac{\pi\,x}{w}\right)\exp\!\left(-\frac{i\,\pi^2\,\hbar\,t}{2\,m\,a\,w}\right)$$

This is simply the ground state wavefunction for an instantaneous infinite square well of width w, with a phase factor, and is thus consistent with the adiabatic approximation.