MAU34404: Quantum Mechanics II
Homework 3 due 25/02/2022

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JS Theoretical Physics

The referenced question from Weinberg's book only includes the third part of the question, and thus assumes from the start that the perturbation transitions from state n to state m in a time interval $t \gg T$. This assignment, however, only assumes this for the third part of the homework, yet I don't think the first two parts make much sense without these assumptions. I have thus done this homework question twice; on the left I have not used the assumptions for the first two parts, and on the right I have, which I think makes more sense in light of the context of the referenced question. I apologise for the confusion and hassle this might cause.

1.

$$
\begin{aligned}
c_a(t) &= c_a(0) - \frac{i}{\hbar}\sum_b c_b(0)\int_0^t dt'\,\delta H_{ab}(t')\,e^{\frac{i\,t'}{\hbar}\,(E_a-E_b)} \\
&= c_a(0) - \frac{i}{\hbar}\sum_b c_b(0)\int_0^t dt'\left\langle \psi_a \,\middle|\, U\,e^{-\frac{t'}{T}} \,\middle|\, \psi_b\right\rangle e^{\frac{i\,t'}{\hbar}\,(E_a-E_b)} \\
&= c_a(0) - \frac{i}{\hbar}\sum_b c_b(0)\int_0^t dt'\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle\, e^{t'\left(\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}\right)} \\
&= c_a(0) - \frac{i}{\hbar}\sum_b c_b(0)\,\frac{\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}}\,e^{t'\left(\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}\right)}\Bigg|_0^t \\
c_a(t) &= c_a(0) - \sum_b c_b(0)\,\frac{i\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{i\,(E_a-E_b)-\frac{\hbar}{T}}\left(e^{t\left(\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}\right)}-1\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
\Gamma(1\to a) &\equiv \frac{|c_a(t)|^2}{t} \\
\Gamma(1\to a) &= \frac{1}{t}\left|c_a(0) - \sum_b c_b(0)\,\frac{i\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{i\,(E_a-E_b)-\frac{\hbar}{T}}\left(e^{t\left(\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}\right)}-1\right)\right|^2
\end{aligned}
$$

1.

$$
c_a(t) = c_a(0) - \frac{i}{\hbar}\sum_b c_b(0)\int_0^t dt'\,\delta H_{ab}(t')\,e^{\frac{i\,t'}{\hbar}\,(E_a-E_b)}
$$

Assuming that at $t = 0$ the state is described entirely by Ψ_n, then we have $c_n(0) = 1$ and $c_a(0) = 0$ for all $a \neq n$. Thus we have

$$
\begin{aligned}
c_m(t) &= -\frac{i}{\hbar}\int_0^t dt'\left\langle \psi_a \,\middle|\, U\,e^{-\frac{t'}{T}} \,\middle|\, \psi_b\right\rangle e^{\frac{i\,t'}{\hbar}\,(E_m-E_n)} \\
&= -\frac{i}{\hbar}\int_0^t dt'\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle\, e^{t'\left(\frac{i}{\hbar}(E_m-E_n)-\frac{1}{T}\right)} \\
&= -\frac{i}{\hbar}\,\frac{\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{\frac{i}{\hbar}(E_a-E_b)-\frac{1}{T}}\,e^{t'\left(\frac{i}{\hbar}(E_m-E_n)-\frac{1}{T}\right)}\Bigg|_0^t \\
c_m(t) &\approx \frac{i\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{i\,(E_m-E_n)-\frac{\hbar}{T}} \qquad (t \gg T)
\end{aligned}
$$

2.

$$
\begin{aligned}
\Gamma(n\to m) &\equiv \frac{|c_m(t)|^2}{t} \\
&= \frac{1}{t}\left|\frac{i\,\langle \psi_a \,|\, U \,|\, \psi_b\rangle}{i\,(E_m-E_n)-\frac{\hbar}{T}}\right|^2 \\
\Gamma(n\to m) &= \frac{1}{t}\,\frac{\langle \psi_a \,|\, U \,|\, \psi_b\rangle^2}{(E_m-E_n)^2+\frac{\hbar^2}{T^2}}
\end{aligned}
$$

3.

$$
P(m) = |c_m(t)|^2 = \frac{\langle \psi_a \,|\, U \,|\, \psi_b\rangle^2}{(E_m-E_n)^2+\frac{\hbar^2}{T^2}}
$$