MAU34404: Quantum Mechanics II
Homework 2 due 18/02/2022

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Problem 1

1.

The quantisation condition based on the WKB approximation is

$$\int_{a_E}^{b_E} k(x)\,dx = \left(n + \frac{1}{2}\right)\pi,$$

where a_E and b_E are the left and right turning points of the system, i.e. where $U(a_E) = U(b_E) = E$, n is an integer, and $k(x)$ is given by

$$k(x) \equiv \sqrt{\frac{2\,\mu}{\hbar^2}\,(E - U(x))}$$

for $E \geq U(x)$, where E is the energy of the system, $U(x)$ is the potential energy, and μ is the reduced mass. This formula implies that there is a quantisation condition on the value of the integral of $k(x)$, and thus that there is a quantisation condition on the energy for a given potential energy of the system.

2.

$$U(x) = \frac{\mu\,\omega^2\,x^2}{2} \qquad\qquad b_E = -a_E \equiv a$$

$$\begin{aligned} E &= U(a) = U(-a) \\ &= \frac{\mu\,\omega^2\,a^2}{2} \end{aligned}$$

$$\begin{aligned} \left(n + \frac{1}{2}\right)\pi &= \int_{a_E}^{b_E} k(x)\,dx \\ &= \int_{-a}^{a} dx\,\sqrt{\frac{2\,\mu}{\hbar^2}\left(\frac{\mu\,\omega^2\,a^2}{2} - \frac{\mu\,\omega^2\,x^2}{2}\right)} \\ &= \frac{\mu\,\omega}{\hbar}\int_{-a}^{a} dx\,\sqrt{a^2 - x^2} \\ &= \frac{\mu\,\omega}{\hbar}\,(\text{area of semicircle of radius } a) \\ &= \frac{\mu\,\omega}{\hbar}\,\frac{\pi\,a^2}{2} \\ &= \frac{\mu\,\omega^2\,a^2}{2}\,\frac{\pi}{\hbar\,\omega} \\ &= \frac{E\,\pi}{\hbar\,\omega} \\ \implies E &= \hbar\,\omega\left(n + \frac{1}{2}\right), \text{ as is the case for the harmonic oscillator} \end{aligned}$$

3.

Since the given potential $V(r)$ is bounded and spherically symmetric, we can use the expression

$$\int_0^{b_E} k(r)\,dr = \left(n - \frac{1}{4}\right)\pi,$$

where b_E is the sole turning point of the system, n is a positive integer, and $k(r)$ is given as before (with $U(x) \to V(r)$).

$$\begin{aligned} E &= V(b_E) \\ &= -V_0\,e^{-\frac{b_E}{R}} \end{aligned}$$

$$\begin{aligned}
\left(n-\frac{1}{4}\right)\pi &= \int_0^{b_E} k(r)\,dr \\
&= \int_0^{b_E} dr\,\sqrt{\frac{2\,\mu}{\hbar^2}\left(-V_0\,e^{-\frac{b_E}{R}} + V_0\,e^{-\frac{r}{R}}\right)} \\
&= \frac{\sqrt{2\,\mu\,V_0}}{\hbar}\int_0^{b_E} dr\,\sqrt{e^{-\frac{r}{R}} - e^{-\frac{b_E}{R}}} \\
&= \frac{e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\int_0^{b_E} dr\,\sqrt{e^{\frac{b_E - r}{R}} - 1} \\
&= \frac{R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\int_0^{\frac{b_E}{R}} du\,\sqrt{e^u - 1} && \left(u = \tfrac{b_E - r}{R} \implies dr = -R\,dr\right) \\
&= \frac{R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\int_1^{e^{\frac{b_E}{R}}} dv\,\frac{\sqrt{v-1}}{v} && \left(v = e^u \implies du = v^{-1}\,dv\right) \\
&= \frac{R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\int_0^{\sqrt{e^{\frac{b_E}{R}}-1}} dw\,\frac{2\,w^2}{w^2+1} && \left(w = \sqrt{v-1} \implies dv = 2\,w\,dw\right) \\
&= \frac{2\,R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\int_0^{\sqrt{e^{\frac{b_E}{R}}-1}} dw\,\left(1 - \frac{1}{w^2+1}\right) \\
&= \frac{2\,R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\left(w - \tan^{-1} w\right)\Big|_0^{\sqrt{e^{\frac{b_E}{R}}-1}} \\
&= \frac{2\,R\,e^{-\frac{b_E}{2\,R}}\sqrt{2\,\mu\,V_0}}{\hbar}\left(\sqrt{e^{\frac{b_E}{R}} - 1} - \tan^{-1}\sqrt{e^{\frac{b_E}{R}} - 1}\right) \\
&= \frac{2\,R\sqrt{2\,\mu\,V_0}}{\hbar}\sqrt{-\frac{E}{V_0}}\left(\sqrt{-\frac{V_0}{E} - 1} - \tan^{-1}\sqrt{-\frac{V_0}{E} - 1}\right) && \left(E = -V_0\,e^{-\frac{b_E}{R}}\right) \\
\left(n-\frac{1}{4}\right)\pi &= \frac{2\,R\sqrt{2\,\mu}}{\hbar}\left(\sqrt{V_0 + E} - \sqrt{-E}\tan^{-1}\sqrt{-\frac{V_0}{E} - 1}\right)
\end{aligned}$$

The solutions to this transcendental equation give an estimate for the energy levels of the given potential.