MAU34404: Quantum Mechanics II
Homework 1 due 11/02/2022

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JS Theoretical Physics

Problem 1

1.

$$\begin{aligned}
\delta H\, \Psi_0 &= \sum_n \langle \Psi_n \,|\, \delta H \,|\, \Psi_0 \rangle\, \Psi_n \\
\langle \Psi_n \,|\, \delta H \,|\, \Psi_0 \rangle &= \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2^n\, n!}}\, H_n\big(x\sqrt{\alpha}\big)\, \Psi_0(x)\, \lambda\, x^4\, \Psi_0(x) \\
&= \frac{\lambda\sqrt{\alpha}}{\sqrt{2^n\, \pi\, n!}} \int_{-\infty}^{\infty} dx\, H_n\big(x\sqrt{\alpha}\big)\, x^4\, e^{-\alpha x^2}
\end{aligned}$$

If n is odd, then the resulting Hermite polynomial will be a polynomial only in odd orders of x. This will result in a sum of integrals of odd functions on infinite bounds, which will vanish. Thus the only non-vanishing cases are when n is even. Since Ψ_4 can be expressed as Ψ_0 times a polynomial in x of degree 4, then $\delta H\, \Psi_0 = \lambda\, x^4\, \Psi_0$ can be found by rearranging and scaling this expression of Ψ_4. Thus the only non-vanishing cases are $n = 0, 2, 4$. We can find the coefficients of $\Psi_{0,2,4}$ by the method above or by directly integrating, the latter of which is done below.

$$\begin{aligned}
\langle \Psi_0 \,|\, \delta H \,|\, \Psi_0 \rangle &= \frac{\lambda\sqrt{\alpha}}{\sqrt{2^0\, \pi\, 0!}} \int_{-\infty}^{\infty} dx\, H_0\big(x\sqrt{\alpha}\big)\, x^4\, e^{-\alpha x^2} \\
&= \lambda \sqrt{\frac{\alpha}{\pi}}\, \frac{\partial^2}{\partial \alpha^2} \left(\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \right) \\
&= \lambda \sqrt{\frac{\alpha}{\pi}}\, \frac{\partial^2}{\partial \alpha^2} \left(\sqrt{\frac{\pi}{\alpha}} \right) \\
&= \lambda \sqrt{\frac{\alpha}{\pi}}\, \sqrt{\pi} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \alpha^{-\frac{5}{2}} \\
&= \frac{3}{4}\, \frac{\lambda}{\alpha^2}
\end{aligned}$$

$$\begin{aligned}
\langle \Psi_2 \,|\, \delta H \,|\, \Psi_0 \rangle &= \frac{\lambda\sqrt{\alpha}}{\sqrt{2^2\, \pi\, 2!}} \int_{-\infty}^{\infty} dx\, H_2\big(x\sqrt{\alpha}\big)\, x^4\, e^{-\alpha x^2} \\
&= \frac{\lambda}{2} \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} dx\, \left(4\,\alpha\, x^2 - 2\right) x^4\, e^{-\alpha x^2} \\
&= \lambda \sqrt{\frac{\alpha}{2\pi}} \left(-2\,\alpha\, \frac{\partial^3}{\partial \alpha^3} \left(\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \right) - \frac{\partial^2}{\partial \alpha^2} \left(\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \right) \right) \\
&= \lambda \sqrt{\frac{\alpha}{2\pi}} \left(-2\,\alpha\, \frac{\partial^3}{\partial \alpha^3} \left(\sqrt{\frac{\pi}{\alpha}} \right) - \frac{\partial^2}{\partial \alpha^2} \left(\sqrt{\frac{\pi}{\alpha}} \right) \right) \\
&= \lambda \sqrt{\frac{\alpha}{2\pi}} \left(-2\,\alpha\, \sqrt{\pi} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \alpha^{-\frac{7}{2}} - \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \alpha^{-\frac{5}{2}} \right) \\
&= \frac{3}{\sqrt{2}}\, \frac{\lambda}{\alpha^2}
\end{aligned}$$

$$
\begin{aligned}
\langle \Psi_4 \,|\, \delta H \,|\, \Psi_0 \rangle &= \frac{\lambda \sqrt{\alpha}}{\sqrt{2^4\,\pi\,4!}} \int_{-\infty}^{\infty} dx\, H_4\big(x\sqrt{\alpha}\big)\, x^4\, e^{-\alpha\, x^2} \\
&= \frac{\lambda}{48} \sqrt{\frac{6\,\alpha}{\pi}} \int_{-\infty}^{\infty} dx\, \big(16\,\alpha^2 x^4 - 48\,\alpha\, x^2 + 12\big)\, x^4\, e^{-\alpha\, x^2} \\
&= \frac{\lambda}{12} \sqrt{\frac{6\,\alpha}{\pi}} \left(4\,\alpha^2 \frac{\partial^4}{\partial\alpha^4}\left(\int_{-\infty}^{\infty} dx\, e^{-\alpha\, x^2}\right) + 12\,\alpha \frac{\partial^3}{\partial\alpha^3}\left(\int_{-\infty}^{\infty} dx\, e^{-\alpha\, x^2}\right) + 3\,\frac{\partial^2}{\partial\alpha^2}\left(\int_{-\infty}^{\infty} dx\, e^{-\alpha\, x^2}\right)\right) \\
&= \frac{\lambda}{12} \sqrt{\frac{6\,\alpha}{\pi}} \left(4\,\alpha^2 \frac{\partial^4}{\partial\alpha^4}\left(\sqrt{\frac{\pi}{\alpha}}\right) + 12\,\alpha \frac{\partial^3}{\partial\alpha^3}\left(\sqrt{\frac{\pi}{\alpha}}\right) + 3\,\frac{\partial^2}{\partial\alpha^2}\left(\sqrt{\frac{\pi}{\alpha}}\right)\right) \\
&= \frac{\lambda}{12} \sqrt{\frac{6\,\alpha}{\pi}} \left(4\,\alpha^2 \sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\alpha^{-\frac{9}{2}}\right. \\
&\qquad\qquad \left. + \,12\,\alpha\sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\alpha^{-\frac{7}{2}} + 3\sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\alpha^{-\frac{5}{2}}\right) \\
&= \frac{\sqrt{6}}{2}\,\frac{\lambda}{\alpha^2}
\end{aligned}
$$

$$
\implies \delta H\, \Psi_0 = \frac{\lambda}{\alpha^2}\left(\frac{3}{4}\,\Psi_0 + \frac{3}{\sqrt{2}}\,\Psi_2 + \frac{\sqrt{6}}{2}\,\Psi_4\right)
$$

2.

$$
\begin{aligned}
\delta_1 \Psi_0 &= \sum_{n\neq 0} \frac{\langle \Psi_n \,|\, \delta H \,|\, \Psi_0\rangle}{E_0 - E_n}\, \Psi_n \\
&= \sum_{n=2,\,4} \frac{\langle \Psi_n \,|\, \delta H \,|\, \Psi_0\rangle}{-n\,\hbar\,\omega}\, \Psi_n \\
&= -\frac{\lambda}{\hbar\,\omega\,\alpha^2}\left(\frac{3}{2\sqrt{2}}\,\Psi_2 + \frac{\sqrt{6}}{8}\,\Psi_4\right) \\
\delta_1 \Psi_0 &= -\frac{\lambda}{2\,\hbar\,\omega\,\alpha^2}\left(\frac{3}{\sqrt{2}}\,\Psi_2 + \frac{\sqrt{6}}{4}\,\Psi_4\right)
\end{aligned}
$$

3.

$$
\delta_2 E_0 = \sum_{n\neq 0} \frac{\left|\langle \Psi_n \,|\, \delta_1 H \,|\, \Psi_0\rangle\right|^2}{E_0 - E_n} + \langle \Psi_0 \,|\, \delta_2 H \,|\, \Psi_0\rangle
$$

$$
\delta H = \lambda\, x^4 \implies \begin{cases} \delta_1 H = \delta H \\ \delta_2 H = 0 \end{cases}
$$

$$
\begin{aligned}
\implies \delta_2 E_0 &= \sum_{n=2,\,4} \frac{\left|\langle \Psi_n \,|\, \delta H \,|\, \Psi_0\rangle\right|^2}{-n\,\hbar\,\omega} \\
&= -\frac{1}{2\,\hbar\,\omega}\,\frac{18\,\lambda^2}{4\,\alpha^4} - \frac{1}{4\,\hbar\,\omega}\,\frac{6\,\lambda^2}{4\,\alpha^4} \\
&= -\frac{21\,\lambda^2}{8\,\hbar\,\omega\,\alpha^4}
\end{aligned}
$$

The ratio $\delta_1 E_0 : \delta_2 E_0$ between the first and second order changes in E_0 is $\frac{3\,\lambda}{4\,\alpha^2} : -\frac{21\,\lambda^2}{8\,\hbar\,\omega\,\alpha^4} = 1 : -\frac{7\,\lambda}{2\,\hbar\,\omega\,\alpha^2}$.