MAU23206: Calculus on Manifolds
Homework 9 due 08/04/2022

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Problem 1

a)

If we define $\gamma : \mathbb{R} \to \mathbb{R}^n$ as $\gamma(t) = \begin{cases} (\cos t, \sin t, 0, \ldots, 0) & t \in (0, 2\pi) \\ 0 & \text{otherwise} \end{cases}$ then M is simply the unit circle about the origin without the point (1,0) embedded in $\mathbb{R}^n$, which is trivially a 1-manifold without boundary.

b)

Consider $\alpha : (0, 2\pi) \to \mathbb{R}^n$ defined as $\alpha(t) = \gamma(t)$. Then α is a coordinate patch that covers M entirely. Thus $\{\alpha\}$ is a sufficient orientation for M, and so M is orientable.

c)

$$\begin{aligned}\int_M \omega &\equiv \sum_{i=1}^{l} \int_{U_i} \alpha_i^*(\varphi_i \omega) \\ &= \sum_{i=1}^{1} \int_{(0,2\pi)} \alpha^*(1 \cdot \omega) \\ &= \int_0^{2\pi} \gamma^* \omega\end{aligned}$$

However, one could also choose $\{\beta\}$ as a sufficient orientation for M, where $\beta : (0, 2\pi) \to \mathbb{R}^n$ is defined as $\beta(t) = (\cos t, -\sin t, 0, \ldots, 0)$, as the patch β also entirely covers M. In this case, we would result in an overall negative sign as a factor of the above answer. Thus, depending on the choice of orientation,

$$\int_M \omega = \pm \int_0^{2\pi} \gamma^* \omega$$

d)

In b) we chose the orientation $\{\alpha\}$, and so as shown in c) there is a positive sign in front of the integral.

Problem 2

a)

Since M takes the form $M = \{x \in U \,|\, f(x) \geq 0\}$ for a smooth function f, then if we can show that $Df(x) \neq 0$ then M is a manifold with boundary, with $\partial M = \{x \in U \,|\, f(x) = 0\}$.

$Df((x, y)) = \begin{pmatrix} y + 2x & x + 2y \end{pmatrix}$. If we have $Df((x, y)) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ then we require $x = y = 0$. However, $(0, 0) \notin M$ as $f((0, 0)) = 0 \ngeq 1$, and so $Df((x, y)) \neq 0$ for all $(x, y) \in M$. Therefore M is a 2-manifold with boundary.

b)

The dimension n of the boundary of M is simply 1, since the dimension of M is 2. Thus, since the induced orientation is simply $(-1)^{n+1}$ times the restricted orientation, we have that the induced and restricted orientations are equivalent. Solving $f(x) = 0$ for x in terms of y leads to the solutions $x_\pm = -\frac{y}{2} \pm \frac{\sqrt{4-3y^2}}{2}$. Since we only consider real solutions, the restriction $4 - 3y^2 > 0$, i.e. $-\frac{2}{\sqrt{3}} < y < \frac{2}{\sqrt{3}}$, is imposed. We can thus define the charts $\alpha_\pm : \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \to \partial M$ by $\alpha_\pm(t) = \left(-\frac{t}{2} \pm \frac{\sqrt{4-3t^2}}{2}, t\right)$. We thus have α_+ and $-\alpha_-$ are positive charts for ∂M.

c)

Using the expression $\int_A F^*\omega = \int_B \omega$ with $A \equiv \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $B \equiv \partial M$, $F \equiv \alpha_+ - \alpha_-$, and $\omega \equiv x\,dy - y\,dx$, we can compute

$$
\begin{aligned}
\int_{\partial M} (x\,dy - y\,dx) &= \int_{\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)} (\alpha_+ - \alpha_-)^*(x\,dy - y\,dx) \\
&= \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} dt \left[\left(-\frac{t}{2} + \frac{\sqrt{4-3t^2}}{2}\right) \cdot 1 - t\left(-\frac{1}{2} - \frac{6t}{4\sqrt{4-3t^2}}\right)\right] \\
&\quad - \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} dt \left[\left(-\frac{t}{2} - \frac{\sqrt{4-3t^2}}{2}\right) \cdot 1 - t\left(-\frac{1}{2} + \frac{6t}{4\sqrt{4-3t^2}}\right)\right] \\
&= \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} dt \left(\sqrt{4-3t^2} + \frac{3t^2}{\sqrt{4-3t^2}}\right) \\
&= \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} dt \frac{4}{\sqrt{4-3t^2}} \\
&= \frac{4}{\sqrt{3}} \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} dt \frac{1}{\sqrt{\frac{4}{3} - t^2}} \\
&= \frac{4}{\sqrt{3}} \sin^{-1}\left(\frac{t}{\frac{2}{\sqrt{3}}}\right)\Bigg|_{t=-\frac{2}{\sqrt{3}}}^{t=\frac{2}{\sqrt{3}}} \\
&= \frac{4\pi}{\sqrt{3}}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
&d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \\
\implies\ &d\omega \wedge \eta = d(\omega \wedge \eta) - (-1)^k \omega \wedge d\eta
\end{aligned}
$$

$$
\begin{aligned}
\implies \int_M d\omega \wedge \eta &= \int_M d(\omega \wedge \eta) - \int_M (-1)^k \omega \wedge d\eta \\
&= \int_{\partial M} \omega \wedge \eta - (-1)^k \int_M \omega \wedge d\eta \\
&= -(-1)^k \int_M \omega \wedge d\eta \qquad \text{(as } \partial M = \emptyset\text{)}
\end{aligned}
$$