MAU23206: Calculus on Manifolds
Homework 8 due 01/04/2022

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Problem 1

Let γ and δ be positive with respect to the orientations of M and N, respectively.

$$\gamma^*\omega_M = a\,dx^1 \wedge \ldots \wedge dx^d \qquad \delta^*\omega_N = b\,dx^1 \wedge \ldots \wedge dx^d \qquad (a, b > 0)$$

$$\begin{aligned}
\implies \omega_N &= \left(\delta^{-1}\right)^* \gamma^* \left(\frac{b}{a}\omega_M\right) \\
&= \left(\gamma \circ \delta^{-1}\right)^* \left(\frac{b}{a}\omega_M\right) \\
\implies F^*\omega_N &= F^* \left(\gamma \circ \delta^{-1}\right)^* \left(\frac{b}{a}\omega_M\right) \\
&= \left(\gamma \circ \delta^{-1} \circ F\right)^* \left(\frac{b}{a}\omega_M\right) \\
\det DF(p) \left(\omega_N\right)_{F(p)} &= \det D\big[\gamma \circ \delta^{-1} \circ F(p)\big] \,\frac{b}{a} \left(\omega_M\right)_{\gamma\circ\delta^{-1}\circ F(p)}
\end{aligned}$$

Since a and b are positive functions by definition, we have that $\det DF(p)$ is positive if and only if $\det D\big[\gamma \circ \delta^{-1} \circ F(p)\big]$ is positive. Denoting $u(p) \equiv \det D\big[\gamma \circ \delta^{-1} \circ F(p)\big]\,\frac{b}{a}$, i.e. $F^*\omega_N = u\,\omega_M$, we then have that F is orientation-preserving if and only if u is positive.

Problem 2

Say n is even. Then we have $\det \alpha_i^+ > 0$ and $\det \alpha_i^- < 0$, and the induced orientation is -1 the restricted orientation. Thus α_i^- is positive in the induced orientation.

Say n is odd. Then we have $\det \alpha_i^+ < 0$ and $\det \alpha_i^- > 0$, and the induced orientation is the restricted orientation. Thus α_i^- is positive in the induced orientation.

Therefore α_i^- is positive in the induced orientation, independent of n.

Problem 3

$$\begin{aligned}
F \text{ orientation-preserving} &\iff \det DF(p) > 0\,\forall\, p \in M \\
&\implies \det[D(\,F|_{\partial M})(p)] > 0\,\forall\, p \in \partial M \\
&\implies F|_{\partial M} \text{ orientation-preserving}
\end{aligned}$$

Now say that ∂M is orientation-preserving, i.e. $\det[D(\,F|_{\partial M})(p)] > 0\,\forall\, p \in \partial M$. Since M and N are connected and $\det DF(p)$ is continuous, then for $\det DF(p)$ to be negative there must be a point $q \in M$ where $\det DF(q) = 0$. F is a diffeomorphism, however, and so $\det DF(q) \neq 0$ for any q. Thus $\det DF(p)$ cannot be negative or zero for any $p \in M$, and so F is orientation-preserving.