MAU23206: Calculus on Manifolds
Homework 7 due 25/03/2022

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

Consider $\alpha_j, \alpha_k \in \{\alpha_i\}$ and $p \in \mathbb{H}^d$ such that $\tau(p) \in U$.

$$\begin{aligned}
\det\left[D\left(\alpha_j^{-1} \circ (\alpha \circ \tau)\right)(p)\right] &= \det\left[D\left(\alpha_j^{-1} \circ \alpha_k \circ \alpha_k^{-1} \circ \alpha \circ \tau\right)(p)\right] \\
&= \det\Big[D\Big(\left(\alpha_k^{-1} \circ \alpha_j\right)^{-1} \circ \left(\alpha_k^{-1} \circ \alpha\right) \circ \tau\Big)(p)\Big] \\
&= \det\Big[D\Big(\left(\alpha_k^{-1} \circ \alpha_j\right)^{-1}\Big)\left(\alpha_k^{-1} \circ \alpha \circ \tau(p)\right) D\left(\alpha_k^{-1} \circ \alpha\right)(\tau(p))\, D\tau(p)\Big] \\
\det\left[D\left(\alpha_j^{-1} \circ (\alpha \circ \tau)\right)(p)\right] &= \det\Big[D\Big(\left(\alpha_k^{-1} \circ \alpha_j\right)^{-1}\Big)\left(\alpha_k^{-1} \circ \alpha \circ \tau(p)\right)\Big] \det\left[D\left(\alpha_k^{-1} \circ \alpha\right)(\tau(p))\right] \det[D\tau(p)]
\end{aligned}$$

By definition, α_j and α_k overlap positively and τ is an orientation-reversing diffeomorphism, and so in the above product the first term is positive and the final term is negative. Thus the sign of $\det\left[D\left(\alpha_j^{-1} \circ (\alpha \circ \tau)\right)(p)\right]$ is positive if and only if the sign of $\det\left[D\left(\alpha_k^{-1} \circ \alpha\right)(\tau(p))\right]$ is negative. Since this is the case for any $\alpha_j, \alpha_k \in \{\alpha_i\}$ and any $p \in \mathbb{H}^d$ such that $\tau(p) \in U$ then this is equivalent to saying that $\alpha \circ \tau$ is a positive patch if and only if α is not a positive patch. By contraposition, we can also say that α is a positive patch if and only if $\alpha \circ \tau$ is not a positive patch. Therefore either α or $\alpha \circ \tau$ is a positive patch.

Problem 2

Say that $(N, \{\alpha_i\})$ is an orientation, and consider $\alpha_j, \alpha_k \in \{\alpha_i\}$ and p such that $\alpha_k(p) \in N$.

$$\begin{aligned}
&\implies \det\left[D\left(\alpha_j^{-1} \circ \alpha_k\right)(p)\right] > 0 \\
&\implies \det\left[D\left(\alpha_j^{-1} \circ f \circ f^{-1}\alpha_k\right)(p)\right] > 0 \\
&\implies \det\Big[D\Big(\left(f^{-1} \circ \alpha_j\right)^{-1} \circ \left(f^{-1} \circ \alpha_k\right)\Big)(p)\Big] > 0
\end{aligned}$$

Since we have $\alpha_k(p) \in N$ then $f^{-1} \circ \alpha_k(p) \in M$. Since $f^{-1} \circ \alpha_j$ and $f^{-1} \circ \alpha_k$ positively overlap for all $\alpha_j, \alpha_k \in \{\alpha_i\}$ then $(M, \left\{f^{-1} \circ \alpha_i\right\})$ is an orientation, and so M is orientable.