MAU23206: Calculus on Manifolds
Homework 6 due 18/03/2022

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

$$\begin{aligned} d\omega &= d(xy\,dx) + d(2z\,dy) - d(y\,dz) \\ &= y\,dx \wedge dx + x\,dy \wedge dx + 2\,dz \wedge dy - dy \wedge dz \\ d\omega &= -x\,dx \wedge dy - 3\,dy \wedge dz \end{aligned}$$

$$\begin{aligned} \alpha^*\omega &= uvu^2\,d(uv) + 2(3u+v)\,d\big(u^2\big) - u^2\,d(3u+v) \\ &= u^3v(v\,du + u\,dv) + (6u+2v)(2u\,du) - u^2(3\,du + dv) \\ &= u^3v^2\,du + u^4v\,dv + 12u^2\,du + 4uv\,du - 3u^2\,du - u^2\,dv \\ \alpha^*\omega &= \big(u^3v^2 + 9u^2 + 4uv\big)\,du + \big(u^4v - u^2\big)\,dv \end{aligned}$$

$$\begin{aligned} \alpha^*(d\omega) &= -uv\,d(uv) \wedge d\big(u^2\big) - 3\,d\big(u^2\big) \wedge d(3u+v) \\ &= -uv(v\,du + u\,dv) \wedge (2u\,du) - 6u\,du \wedge (3\,du + dv) \\ &= -2u^2v(v\,du \wedge du + u\,dv \wedge du) - 6u(3\,du \wedge du + du \wedge dv) \\ &= -2u^3v\,dv \wedge du - 6u\,du \wedge dv \\ \alpha^*(d\omega) &= \big(2u^3v - 6u\big)\,du \wedge dv \end{aligned}$$

$$\begin{aligned} d(\alpha^*\omega) &= d\big[\big(u^3v^2 + 9u^2 + 4uv\big)\,du\big] + 4\big[\big(u^4v - u^2\big)\,dv\big] \\ &= \big(3u^2v^2\,du + 2u^3v\,dv + 18u\,du + 4v\,du + 4u\,dv\big) \wedge du + \big(4u^3v\,du + u^4\,dv - 2u\,du\big) \wedge dv \\ &= \big(2u^3v + 4u\big)\,dv \wedge du + \big(4u^3v - 2u\big)\,du \wedge dv \\ d(\alpha^*\omega) &= \big(2u^3v - 6u\big)\,du \wedge dv \end{aligned}$$

Problem 2

Elements of $\Omega^k(A)$ are of the form

$$\begin{aligned} v_1\,dx + v_2\,dy + v_3\,dz &\in \Omega^1(A), \\ w_1\,dx \wedge dy + w_2\,dy \wedge dz + w_3\,dz \wedge dx &\in \Omega^2(A), \\ c\,dx \wedge dy \wedge dz &\in \Omega^3(A). \end{aligned}$$

These elements are analogous to the elements of $\mathfrak{X}(A)$ and $\mathbb{R}$ of the form

$$\begin{aligned} v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 &\in \mathfrak{X}(A), \\ w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3 &\in \mathfrak{X}(A), \\ c &\in C^\infty(A). \end{aligned}$$

Thus, $\Omega^1(A) \times \Omega^1(A) \to \Omega^2(A)$ is analogous to $\mathfrak{X}(A) \times \mathfrak{X}(A) \to \mathfrak{X}(A)$, and $\Omega^1(A) \times \Omega^2(A) \to \Omega^3(A)$ to $\mathfrak{X}(A) \times \mathfrak{X}(A) \to C^\infty(A)$. These correspond to the cross product $(\times)$ and dot product $(\cdot)$, respectively, as $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$.

Problem 3

For convenience, denote $\{\beta_i\} \equiv \{\beta : U \to V \mid \beta \text{ a positive chart}\}$ and β_j a positive chart.

We first need to show that $\{\beta_i\}$ is an orientation, i.e. that β_j and β_k overlap positively for all $\beta_j, \beta_k \in \{\beta_i\}$.

$$\begin{aligned}
\beta_j^{-1} \circ \beta_k &= \beta_j^{-1} \circ \alpha_l^{-1} \circ \alpha_l \circ \beta_k && (\alpha_l \in \{\alpha_i\}) \\
&= (\alpha_l \circ \beta_j)^{-1} \circ (\alpha_l \circ \beta_k) \\
D\big(\beta_j^{-1} \circ \beta_k\big)(x) &= D\Big((\alpha_l \circ \beta_j)^{-1} \circ (\alpha_l \circ \beta_k)\Big)(x) \\
&= D\Big((\alpha_l \circ \beta_j)^{-1}\Big)(\alpha_l \circ \beta_k(x))\, D(\alpha_l \circ \beta_k)(x) \\
\det\big[D\big(\beta_j^{-1} \circ \beta_k\big)(x)\big] &= \det\Big[D\Big((\alpha_l \circ \beta_j)^{-1}\Big)(\alpha_l \circ \beta_k(x))\, D(\alpha_l \circ \beta_k)(x)\Big] \\
&= \det\Big[D\Big((\alpha_l \circ \beta_j)^{-1}\Big)(\alpha_l \circ \beta_k(x))\Big] \det[D(\alpha_l \circ \beta_k)(x)] \\
&= a \cdot b
\end{aligned}$$

Since α_l overlaps positively with both β_j and β_k, then both a and b are positive, and so their product is positive. We thus have

$$\det\big[D\big(\beta_j^{-1} \circ \beta_k\big)(x)\big] > 0,$$

and so β_j and β_k overlap positively. Since this is true for any elements of $\{\beta_i\}$ then the collection is an orientation.

Next we have to show that $\{\alpha_i\} \subset \{\beta_i\}$. Consider $\alpha_j \in \{\alpha_i\}$. By definition, this overlaps positively with all elements of $\{\alpha_i\}$, including α_j itself, since $\det\big[D\big(\alpha_j^{-1} \circ \alpha_j\big)(x)\big] = 1 > 0$. Thus $\alpha_j \in \{\beta_i\}$. This is true for any $\alpha_j \in \{\alpha_i\}$, and so we have $\{\alpha_i\} \subset \{\beta_i\}$.

Finally we need to show that $\{\beta_i\}$ is maximal, i.e. that it only contains all positive charts that pairwise positively overlap.

$$\begin{aligned}
\gamma^{-1} \circ \beta_j &= \gamma^{-1} \circ \alpha_k \circ \alpha_k^{-1} \circ \beta_j \\
&= \big(\gamma^{-1} \circ \alpha_k\big) \circ \big(\alpha_k^{-1} \circ \beta_j\big) \\
D\big(\gamma^{-1} \circ \beta_j\big)(x) &= D\big(\big(\gamma^{-1} \circ \alpha_k\big) \circ \big(\alpha_k^{-1} \circ \beta_j\big)\big)(x) \\
&= D\big(\gamma^{-1} \circ \alpha_k\big)\big(\alpha_k^{-1} \circ \beta_j(x)\big)\, D\big(\alpha_k^{-1} \circ \beta_j\big)(x) \\
\det\big[D\big(\gamma^{-1} \circ \beta_j\big)(x)\big] &= \det\big[D\big(\gamma^{-1} \circ \alpha_k\big)\big(\alpha_k^{-1} \circ \beta_j(x)\big)\, D\big(\alpha_k^{-1} \circ \beta_j\big)(x)\big] \\
&= \det\big[D\big(\gamma^{-1} \circ \alpha_k\big)\big(\alpha_k^{-1} \circ \beta_j(x)\big)\big] \det\big[D\big(\alpha_k^{-1} \circ \beta_j\big)(x)\big] \\
c &= d \cdot e
\end{aligned}$$

Since each β_j overlaps positively with α_k then e is positive. Thus the sign of c is positive if and only if the sign of e is positive. This is equivalent to saying that γ and β_j positively overlap if and only if γ and α_k positively overlap. Since this is true for all $\beta_j \in \{\beta_i\}$ and $\alpha_k \in \{\alpha_i\}$ we have γ positively overlaps with all elements of $\{\beta_i\}$ if and only if $\gamma \in \{\beta_i\}$. Therefore $\{\beta_i\}$ is maximal.

Problem 4

Consider coordinate patches $\alpha_j, \alpha_k \in \{\alpha_i\}$ and points w, x, y such that $\alpha_j(x) = w = \alpha_k(y)$.

$$
\begin{aligned}
D\Big((\alpha_k \circ \tau)^{-1} \circ (\alpha_j \circ \tau)\Big)\big(\tau^{-1}(x)\big) &= D\big(\tau^{-1} \circ \alpha_k^{-1} \circ \alpha_j \circ \tau\big)\big(\tau^{-1}(x)\big) \\
&= D\big(\beta_k^{-1} \circ \beta_j\big)\big(\tau^{-1}(x)\big) && (\beta_l \equiv \alpha_l \circ \tau) \\
&= D\beta_k^{-1}\big(\beta_j\big(\tau^{-1}(x)\big)\big)\, D\beta_j\big(\tau^{-1}(x)\big) \\
&= D\big(\tau^{-1} \circ \alpha_k^{-1}\big)\big(\alpha_j \circ \tau\big(\tau^{-1}(x)\big)\big)\, D(\alpha_j \circ \tau)\big(\tau^{-1}(x)\big) \\
&= D\tau^{-1}\big(\alpha_k^{-1}(w)\big)\, D\alpha_k^{-1}(w) D\alpha_j\big(\tau\big(\tau^{-1}(x)\big)\big)\, D\tau\big(\tau^{-1}(x)\big) \\
&= D\tau^{-1}(y) D\big(\alpha_k^{-1} \circ \alpha_j\big)(x) D\tau\big(\tau^{-1}(x)\big) \\
\det\Big[D\Big((\alpha_k \circ \tau)^{-1} \circ (\alpha_j \circ \tau)\Big)\big(\tau^{-1}(x)\big)\Big] &= \det\big(D\tau^{-1}(y)\big) \det\big(D\big(\alpha_k^{-1} \circ \alpha_j\big)(x)\big) \det\big(D\tau\big(\tau^{-1}(x)\big)\big) \\
&= \det\big(D\big(\alpha_k^{-1} \circ \alpha_j\big)(x)\big) \\
& \qquad\qquad (\tau \text{ diffeomorphism} \implies \det(D\tau)\det\big(D\tau^{-1}\big) = 1) \\
&> 0 && (\alpha_j, \alpha_k \text{ overlap positively})
\end{aligned}
$$

Thus if $\{\alpha_i\}$ is an orientation of M, then so is $\{\alpha_i \circ \tau\}$.

If $\{\alpha_i\} = \{\alpha_i \circ \tau\}$, then we must have that for any $\alpha_j \in \{\alpha_i\}$ there exists an $\alpha_k \in \{\alpha_i\}$ such that $\alpha_j \circ \tau = \alpha_k$. Since τ is a diffeomorphism then $\alpha_j \circ \tau = \alpha_k \circ \tau \iff j = k$, which supports the imposed condition. The two orientations are different when this condition is not met.