MAU23206: Calculus on Manifolds
Homework 5 due 04/03/2022

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Problem 1

$$\begin{aligned}
d\omega &= d(xy\,dx) + d(3\,dy) - d(yz\,dz)) \\
&= y\,dx \wedge dx + x\,dy \wedge dx + 0 - z\,dy \wedge dz - y\,dz \wedge dz \\
&= -x\,dx \wedge dy - z\,dy \wedge dz \\
d(d\omega) &= -d(x\,dx \wedge dy) - d(z\,dy \wedge dz) \\
&= -dx \wedge dx \wedge dy - d \wedge dy \wedge dz \\
\implies d(d\omega) &= 0
\end{aligned}$$

$$\begin{aligned}
\omega \wedge \eta =& \ (xy\,dx + 3\,dy - yz\,dz) \wedge \left(x\,dx - yz^2\,dy + 2x\,dz\right) \\
=& \ x^2y\,dx \wedge dx - xy^2z^2\,dx \wedge dy + 2x^2y\,dx \wedge dz \\
& + 3x\,dy \wedge dx - 3yz^2\,dy \wedge dy + 6x\,dy \wedge dz \\
& - xyz\,dz \wedge dx + y^2z^3\,dz \wedge dy - 2xyz\,dz \wedge dz \\
=& \ \left(-xy^2z^2 - 3x\right) dx \wedge dy + \left(6x - y^2z^3\right) dy \wedge dz + \left(-2x^2y - xyz\right) dz \wedge dx \\
d(\omega \wedge \eta) =& \ d\left(\left(-xy^2z^2 - 3x\right) dx \wedge dy\right) + d\left(\left(6x - y^2z^3\right) dy \wedge dz\right) + d\left(\left(-2x^2y - xyz\right) dz \wedge dx\right) \\
=& \ \left(-y^2z^2\,dx - 2xyz^2\,dy - 2xy^2z\,dz - 3\,dx\right) \wedge dx \wedge dy \\
& + \left(6\,dx - 2yz^3\,dy - 3y^2z^2\,dz\right) \wedge dy \wedge dz \\
& + \left(-4xy\,dx - 2x^2\,dy - yz\,dx - xz\,dy - xy\,dz\right) \wedge dz \wedge dx \\
=& \ \left(-2xy^2z + 6 - 2x^2 - xz\right) dx \wedge dy \wedge dz
\end{aligned}$$

$$\begin{aligned}
d\omega &= -x\,dx \wedge dy - z\,dy \wedge dz \\
d\omega \wedge \eta &= (-x\,dx \wedge dy - z\,dy \wedge dz) \wedge \left(x\,dx - yz^2\,dy + 2x\,dz\right) \\
&= -2x^2\,dx \wedge dy \wedge dz - xz\,dy \wedge dz \wedge dx \\
&= \left(-2x^2 - xz\right) dx \wedge dy \wedge dz
\end{aligned}$$

$$\begin{aligned}
d\eta &= 2yz\,dy \wedge dz - 2\,dz \wedge dx \\
\omega \wedge d\eta &= (xy\,dx + 3\,dy - yz\,dz) \wedge (2yz\,dy \wedge dz - 2\,dz \wedge dx) \\
&= 2xy^2z\,dx \wedge dy \wedge dz - 6\,dy \wedge dz \wedge dx \\
&= \left(2xy^2z - 6\right) dx \wedge dy \wedge dz
\end{aligned}$$

$$\begin{aligned}
d\omega \wedge \eta - \omega \wedge d\eta &= \left(-2x^2 - xz - 2xy^2z + 6\right) dx \wedge dy \wedge dz \\
\implies d(\omega \wedge \eta) &= d\omega \wedge \eta - \omega \wedge d\eta
\end{aligned}$$

Problem 2

Let $p \in M$ and $v \in T_pM$ be given. Define $g(t) = f(p + t\,v)$. We thus have that $g'(t) = f'(p + t\,v) = Df(p + t\,v) \cdot v$ by the chain rule. Since f has an extremum at p, then since $g(0) = f(p)$ we must also have that g has an extremum at 0. Therefore $g'(0) = Df(p) \cdot v = df_p(v) = 0$, and so df_p is simply zero.

This argument does not hold for a manifold with boundary, however, since the derivative of f at a boundary point is not defined, and so we cannot say the same is true if M is a manifold with boundary.

Problem 3

a)

$$
\begin{aligned}
d\eta &= d(F^*\omega) \\
&= F^* d\omega \\
&= F^*(2\,dx \wedge dy)
\end{aligned}
$$

Thus we have $d\eta \in \Omega^2\big(T_pS^1\big)$. However, the tangent plane at any point on S^1 is simply a line and thus one-dimensional. Thus $d\eta$ is a 2-form defined on a 1-dimensional vector space, and so must be 0.

b)

$$
\begin{aligned}
\omega_p(v) &= (p_1\,dy - p_2\,dx)((v_1, v_2)) \\
&= \Big(p_1\Psi^{(2)} - p_2\Psi^{(1)}\Big)(v_1e_1 + v_2e_2) \\
&= v_1\Big[p_1\Psi^{(2)}(e_1) - p_2\Psi^{(1)}(e_1)\Big] + v_2\Big[p_1\Psi^{(2)}(e_2) - p_2\Psi^{(1)}(e_2)\Big] \\
&= -v_1p_2 + v_2p_1
\end{aligned}
$$

$$
\begin{aligned}
\eta_p(v) &= (F^*\omega)_p(v) \\
&= (DF(p))^*\omega_{F(p)}(v) \\
&= \omega_{F(p)}(DF(p)v) \\
&= \omega_p(v) && \text{(since } F \text{ is an inclusion)} \\
&= -v_1p_2 + v_2p_1
\end{aligned}
$$

If we had η_p was always zero then we would need $p_1 = p_2 = 0$. However, since $p \in S^1$ then p_1 and p_2 cannot both be 0, therefore η_p is non-zero.

c)

Since η is simply a restriction of ω to be a tensor on S^1 as opposed to $\mathbb{R}^2$, then it is sufficient to show that there is no function $f \in \Omega^0\big(S^1\big)$ such that $df = \omega$.

Say there exists a function $f \in \Omega^0\big(S^1\big)$ such that $df = \omega$. Then we have

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \\
\omega &= -y\,dx + x\,dy
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= -y \\
f &= -xy + c_1(y)
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{\partial f}{\partial y} &= x \\
f &= xy + c_2(x)
\end{aligned}
$$

We thus require that $c_1(y) - c_2(x) = 2xy$, which is a contradiction, and so we can conclude that no such function exists.

d)

$$\begin{aligned}
(\exp^* \eta)_t(x) &= ((\exp^* \circ F^*)\omega)_t(x) \\
&= ((F \circ \exp)^*\omega)_t(x) && \text{(proven in Problem 4)} \\
&= (\exp^* \omega)_t(x) && \text{(since } F \text{ is an inclusion)} \\
&= (D\exp(t))^*\omega_{\exp(t)}(x) \\
&= \omega_{\exp(t)}(D\exp(t)x) \\
&= \omega_{(\cos(t),\sin(t))}((-x\sin(t), x\cos(t))) \\
&= x\sin(t)\sin(t) + x\cos(t)\cos(t) \\
&= x \\
\implies \exp^* \eta = dx
\end{aligned}$$

Clearly, if we have $f = x \in \Omega^0(\mathbb{R})$ then $df = dx = \exp^* \eta$, and so $\exp^* \eta$ is df.

Problem 4

$$\begin{aligned}
(F^*\omega)_p(v_1, \ldots, v_k) &= (DF(p))^*\omega_{F(p)}(v_1, \ldots, v_k) \\
&= \omega_{F(p)}(DF(p)v_1, \ldots, DF(p)v_k)
\end{aligned}$$

a)

$$\begin{aligned}
((\mathrm{id}_M)^*\omega)_p(v_1, \ldots, v_k) &= \omega_{F(p)}(D(\mathrm{id}_M)(p)v_1, \ldots, D(\mathrm{id}_M)(p)v_k) \\
&= \omega_{F(p)}(v_1, \ldots, v_k) \\
&= (\mathrm{id}_{\Omega^k(M)}\omega)_p(v_1, \ldots, v_k) \\
\implies (\mathrm{id}_M)^* = \mathrm{id}_{\Omega^k(M)}
\end{aligned}$$

b)

$$\begin{aligned}
((f \circ g)^*\omega)_p(v_1, \ldots, v_k) &= ((D(f \circ g))(p))^*\omega_{(f\circ g)(p)}(v_1, \ldots, v_k) \\
&= (Df(g(p)) \cdot Dg(p))^*\omega_{f(g(p))}(v_1, \ldots, v_k) \\
&= \omega_{f(g(p))}(Df(g(p)) \cdot Dg(p)v_1, \ldots, Df(g(p)) \cdot Dg(p)v_k) \\
&= (Df(g(p)))^*\omega_{f(g(p))}(Dg(p)v_1, \ldots, Dg(p)v_k) \\
&= (f^*\omega)_{g(p)}(Dg(p)v_1, \ldots, Dg(p)v_k) \\
&= (Dg(p))^*(f^*\omega)_{g(p)}(v_1, \ldots, v_k) \\
&= (g^*(f^*\omega))_p(v_1, \ldots, v_k) \\
&= ((g^* \circ f^*)\omega)_p(v_1, \ldots, v_k) \\
\implies (f \circ g)^* = g^* \circ f^*
\end{aligned}$$