MAU23206: Calculus on Manifolds
Homework 4 due 25/02/2022

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

a)

$$\begin{aligned} f(x,y) &= x_1\,y_2 - x_2\,y_1 + x_1\,y_1 \\ f(y,x) &= y_1\,x_2 - y_2\,x_1 + y_1\,x_1 \\ &= -x_1\,y_2 + x_2\,y_1 + x_1\,y_1 \\ &\neq -f(x,y) \implies f \text{ is not alternating} \end{aligned}$$

b)

g is clearly a tensor since each component of x and y are mapped linearly.

$$\begin{aligned} g(x,y) &= x_1\,y_3 - x_3\,y_1 \\ g(y,x) &= y_1\,x_3 - y_3\,x_1 \\ &= -\left(x_1\,y_3 - x_3\,y_1\right) \\ &= -g(x,y) \implies g \text{ is alternating} \end{aligned}$$

$$\begin{aligned} 2!\,g &= Ag \\ 2\,g &= A(e^1 \otimes e^3 - e^3 \otimes e^1) \\ &= e^1 \wedge e^3 - e^3 \wedge e^1 \\ &= 2\,e^1 \wedge e^3 \\ \implies \quad & g = e^1 \wedge e^3 \end{aligned}$$

c)

h is clearly not a tensor, as the maps of x and y are not linear, and so h is not an alternating tensor.

Problem 2

a)

$$\begin{aligned} F &= 5\,e^3 \otimes e^5 \otimes e^2 - 3\,e^1 \otimes e^2 \otimes e^2 \\ AF &= 5\,e^3 \wedge e^5 \wedge e^2 - 3\,e^1 \wedge e^2 \wedge e^2 \\ AF &= 5\,e^3 \wedge e^5 \wedge e^2 \end{aligned}$$

$$\begin{aligned} G &= e^1 \otimes e^5 - e^4 \otimes e^5 \\ AG &= e^1 \wedge e^5 - e^4 \wedge e^5 \end{aligned}$$

b)

$$\begin{aligned} AF(x,y,z) &= 5\,e^3\wedge e^5\wedge e^2(x,y,z) \\ &= 5\begin{vmatrix} x_3 & y_3 & z_3 \\ x_5 & y_5 & z_5 \\ x_2 & y_2 & z_2 \end{vmatrix} \\ AF(x,y,z) &= 5\,[x_3\,(y_5\,z_2 - y_2\,z_5) + y_3\,(x_2\,z_5 - x_5\,z_2) + z_3\,(x_5\,y_2 - x_2\,y_5)] \end{aligned}$$

$$\begin{aligned} AG(x,y) &= (e^1\wedge e^5 - e^4\wedge e^5)(x,y) \\ &= \begin{vmatrix} x_1 & y_1 \\ x_5 & y_5 \end{vmatrix} - \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \\ &= x_1\,y_5 - x_5\,y_1 - x_4\,y_5 + x_5\,y_4 \\ AG(x,y) &= (y_4 - y_1)\,x_5 + (x_1 - x_4)\,y_5 \end{aligned}$$

$$\begin{aligned} (AF\wedge AG)(v,w,x,y,z) &= (5\,e^3\wedge e^5\wedge e^2)\wedge(e^1\wedge e^5 - e^4\wedge e^5)(v,w,x,y,z) \\ &= 5\,(e^3\wedge e^5\wedge e^2\wedge e^1\wedge e^5 - e^3\wedge e^5\wedge e^2\wedge e^4\wedge e^5)(v,w,x,y,z) \\ (AF\wedge AG)(v,w,x,y,z) &= 0 \end{aligned}$$

Problem 3

$$\begin{aligned} AG(\mathbf{v}_1,\dots,\mathbf{v}_k) &= \sum_\sigma (\operatorname{sgn}\sigma)\,G^\sigma(\mathbf{v}_1,\dots,\mathbf{v}_k) \\ &= G(\mathbf{v}_1,\dots,\mathbf{v}_k)\sum_\sigma \operatorname{sgn}\sigma && (G \text{ symmetric i.e. } G^\sigma = G) \\ &= G(\mathbf{v}_1,\dots,\mathbf{v}_k)\sum_\sigma \operatorname{sgn}(e\circ\sigma) && (\text{since } \sigma\to e\circ\sigma\in S_k \text{ isomorphism}) \\ &= G(\mathbf{v}_1,\dots,\mathbf{v}_k)\sum_\sigma (-\operatorname{sgn}\sigma) && (\text{since } e \text{ inversion}) \\ \implies \sum_\sigma \operatorname{sgn}\sigma &= -\sum_\sigma \operatorname{sgn}\sigma \\ \implies AG(\mathbf{v}_1,\dots,\mathbf{v}_k) &= 0 \end{aligned}$$

Let us show that the converse is not true. Consider $G(x,y,z) = x_1\,y_1 + z_1$. We then have

$$\begin{aligned} AG(x,y,z) &= \sum_\sigma (\operatorname{sgn}\sigma)\,G^\sigma(x,y,z) \\ &= G(x,y,z) + G(y,z,x) + G(z,x,y) - G(x,z,y) - G(y,x,z) - G(z,y,x) \\ &= x_1\,y_1 + z_1 + y_1\,z_1 + x_1 + z_1\,x_1 + y_1 - x_1\,z_1 - y_1 - y_1\,x_1 - z_1 - z_1\,y_1 - x_1 \\ &= 0. \end{aligned}$$

Therefore $AG = 0$. G, however, is not symmetric, since $G(x,z,y) = x_1\,z_1 + y_1 \neq x_1\,y_1 + z_1 = G(x,y,z)$, and so the converse does not hold.

Problem 4

i)

$$\begin{aligned} \det(\mathbf{1}_{n\times n})\,f(\mathbf{v}_1,\dots,\mathbf{v}_n) &= (\mathbf{1}_{n\times n}^* f)(\mathbf{v}_1,\dots,\mathbf{v}_n) \\ &= f(\mathbf{1}_{n\times n}\mathbf{v}_1,\dots,\mathbf{1}_{n\times n}\mathbf{v}_n) \\ &= f(\mathbf{v}_1,\dots,\mathbf{v}_n) \\ \implies \det(\mathbf{1}_{n\times n}) &= 1 \end{aligned}$$

ii)

Define $A(a_1, \dots, a_n) \equiv \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}$ to be the $n \times n$ matrix made up of columns $a_1, \dots, a_n$, and $A_i(b) \equiv A(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ to be the $n \times n$ matrix made up of columns $a_1, \dots, a_n$, replacing a_i with b. If we show that ii) is satisfied when only considering a set of unit vectors $\mathbf{e}_i$ with $(\mathbf{e}_i)_j = \delta_i^j$, then any set of vectors can be written as a linear combination of these vectors, and so by the multilinearity of f ii) will be satisfied for any set of n vectors.

$$\begin{aligned}
A_i^*(\alpha\, b + \beta\, c) f(\mathbf{e}_1, \dots, \mathbf{e}_n) &= f(A_i^*(\alpha\, b + \beta\, c)\, \mathbf{e}_1, \dots, A_i^*(\alpha\, b + \beta\, c)\, \mathbf{e}_n) \\
&= f(a_1, \dots, a_{i-1}, \alpha\, b + \beta\, c, a_{i+1}, \dots, a_n) \\
&= \alpha\, f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) + \beta\, f(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n) \\
&= \alpha\, f(A_i^*(b)\, \mathbf{e}_1, \dots, A_i^*(b)\, \mathbf{e}_n) + \beta\, f(A_i^*(c)\, \mathbf{e}_1, \dots, A_i^*(c)\, \mathbf{e}_n) \\
&= \alpha\, A_i^*(b) f(\mathbf{e}_1, \dots, \mathbf{e}_n) + \beta\, A_i^*(c) f(\mathbf{e}_1, \dots, \mathbf{e}_n) \\
&= (\alpha\, A_i^*(b) + \beta\, A_i^*(c)) f(\mathbf{e}_1, \dots, \mathbf{e}_n) + \beta\, A_i^*(c) f(\mathbf{e}_1, \dots, \mathbf{e}_n) \\
\implies \det(A_i(\alpha\, b + \beta\, c)) &= \alpha \det(A_i(b)) + \beta \det(A_i(c))
\end{aligned}$$

iii)

Define $A(a_1, \dots, a_n)$ as before, and $A_{i,j}(b) \equiv A(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n)$ to be the $n \times n$ matrix made up of columns $a_1, \dots, a_n$, replacing both a_i and a_j with b. As before, we only need to satisfy iii) for a set of unit vectors.

$$\begin{aligned}
A_{i,j}(b)^* f(\mathbf{e}_1, \dots, \mathbf{e}_n) &= f(A_{i,j}(b)\, \mathbf{e}_1, \dots, A_{i,j}(b)\, \mathbf{e}_n) \\
&= f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \\
&= -f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \\
&= 0 \\
\implies \det(A_{i,j}(b)) &= 0
\end{aligned}$$

Problem 5

a)

We must first show that $\mathcal{A}^\bullet(V)$ is an algebra by showing that the operation satisfies left and right distributivity and compatability with scalars.

$$\begin{aligned}
\left(\sum_i (f_i + \phi_i)\right) \wedge \left(\sum_j (g_j + \gamma_j)\right) &= \sum_k \left(\sum_{i+j=k} (f_i + \phi_i) \wedge (g_j + \gamma_j)\right) \\
&= \sum_k \left(\sum_{i+j=k} (f_i \wedge (g_j + \gamma_j) + \phi_i \wedge (g_j + \gamma_j))\right) \\
&= \sum_k \left(\sum_{i+j=k} (f_i \wedge g_j + f_i \wedge \gamma_j + \phi_i \wedge g_j + \phi_i \wedge \gamma_j)\right) \\
&= \sum_k \left(\sum_{i+j=k} f_i \wedge g_j\right) + \sum_k \left(\sum_{i+j=k} f_i \wedge \gamma_j\right) \\
&\quad + \sum_k \left(\sum_{i+j=k} \phi_i \wedge g_j\right) + \sum_k \left(\sum_{i+j=k} \phi_i \wedge \gamma_j\right) \\
&\implies \text{left and right distributivity}
\end{aligned}$$

$$
\begin{aligned}
\left(\sum_i c\, f_i\right) \wedge \left(\sum_j g_j\right) &= \sum_k \left(\sum_{i+j=k} c\, f_i \wedge g_i\right) \\
&= c \sum_k \left(\sum_{i+j=k} f_i \wedge g_j\right) \qquad\qquad = \sum_k \left(\sum_{i+j=k} f_i \wedge c\, g_i\right) \\
&= c\left(\left(\sum_i f_i\right) \wedge \left(\sum_j g_j\right)\right) \qquad = \left(\sum_i f_i\right) \wedge \left(\sum_j c\, g_j\right) \\
&\implies \text{compatability with scalars}
\end{aligned}
$$

Thus $\mathcal{A}^\bullet(V)$ is an algebra. We can show the algebra is unital by considering the operation between an element $f_0 + \ldots + f_l$ and a proposed identity $e_0 + \ldots + e_m$.

$$
\begin{aligned}
\left(\sum_i f_i\right) \wedge \left(\sum_j e_j\right) &= \sum_k \left(\sum_{i+j=k} f_i \wedge e_j\right) \\
&= f_0 \wedge e_0 + (f_0 \wedge e_1 + f_1 \wedge e_0) + \ldots + (f_{l-1} \wedge e_m + f_l \wedge e_{m-1}) + f_l \wedge e_m \\
&= f_0 + f_1 + \ldots + f_{l-1} + f_l \\
\implies e_j = e_0\, \delta_j^0&, \text{ i.e. } e_0 \text{ is the only non-vanishing } e_j
\end{aligned}
$$

We thus require that $f_i \wedge e_0 = f_i$ for all $i = 0, \ldots, l$.

$$
\begin{aligned}
f_i \wedge e_0(\mathbf{v}_1, \ldots, \mathbf{v}_{i+0}) &= \frac{1}{i!\, 0!} A(f_i \otimes e_0)(\mathbf{v}_1, \ldots, \mathbf{v}_i) \\
&= \frac{1}{i!} \sum_\sigma (\operatorname{sgn} \sigma)(f_i \otimes e_0)^\sigma(\mathbf{v}_1, \ldots, \mathbf{v}_i) \\
&= \frac{1}{i!} \sum_\sigma (\operatorname{sgn} \sigma)(f_i \otimes e_0)(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(i)}) \\
&= \frac{1}{i!} \sum_\sigma (\operatorname{sgn} \sigma) f_i(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(i)})\, e_0 \\
&= \frac{e_0}{i!} \sum_\sigma (\operatorname{sgn} \sigma) f_i^\sigma(\mathbf{v}_1, \ldots, \mathbf{v}_i) \\
&= \frac{e_0}{i!} A f_i(\mathbf{v}_1, \ldots, \mathbf{v}_i) \\
&= e_0\, f_i(\mathbf{v}_1, \ldots, \mathbf{v}_i) && \text{(since } f \text{ is alternating)}
\end{aligned}
$$

Thus the 0-tensor that maps to 1 is the identity element of $\mathcal{A}^\bullet(V)$, and so $\mathcal{A}^\bullet(V)$ is a unital algebra. Now we must show that it is also an associative algebra by considering the operation between elements $f_0 + \ldots + f_a$, $g_0 + \ldots + g_b$, and $h_0 + \ldots + h_c$.

$$
\begin{aligned}
\left(\left(\sum_i f_i\right) \wedge \left(\sum_j g_j\right)\right) \wedge \left(\sum_k h_k\right) &= \left(\sum_l \left(\sum_{i+j=l} f_i \wedge g_j\right)\right) \wedge \left(\sum_k h_k\right) \\
&= \left(\sum_l \phi_l\right) \wedge \left(\sum_k h_k\right) \quad (\phi_l \equiv \textstyle\sum_{i+j=l} f_i \wedge g_j,\ l = 0, \dots, a+b) \\
&= \sum_m \left(\sum_{l+k=m} \phi_l \wedge h_k\right) \\
&= \phi_0 \wedge h_0 \\
&\quad + \phi_0 \wedge h_1 + \phi_1 \wedge h_0 \\
&\quad + \dots \\
&\quad + \phi_{a+b-1} \wedge h_c + \phi_{a+b} \wedge h_{c-1} \\
&\quad + \phi_{a+b} \wedge h_c \\
&= (f_0 \wedge g_0) \wedge h_0 \\
&\quad + (f_0 \wedge g_0) \wedge h_1 + (f_0 \wedge g_1 + f_1 \wedge g_0) \wedge h_0 \\
&\quad + \dots \\
&\quad + (f_{a-1} \wedge g_b + f_a \wedge g_{b-1}) \wedge h_c + (f_a \wedge g_b) \wedge h_{c-1} \\
&\quad + (f_a \wedge g_b) \wedge h_c \\
&= f_0 \wedge (g_0 \wedge h_0) \\
&\quad + f_0 \wedge (g_0 \wedge h_1 + g_1 \wedge h_0) + f_1 \wedge (g_0 \wedge h_0) \\
&\quad + \dots \\
&\quad + f_{a-1} \wedge (g_b \wedge h_c) + f_a \wedge (g_{b-1} \wedge h_c + g_b \wedge h_{c-1}) \\
&\quad + f_a \wedge (g_b \wedge h_c) \\
&\qquad\qquad (\text{using the associativity and distributivity of } \wedge) \\
&= f_0 \wedge \gamma_0 \\
&\quad + f_0 \wedge \gamma_1 + f_1 \wedge \gamma_0 \\
&\quad + \dots \\
&\quad + f_{a-1} \wedge \gamma_{b+c} + f_a \wedge \gamma_{b+c-1} \\
&\quad + f_a \wedge \gamma_{b+c} \qquad (\gamma_p \equiv \textstyle\sum_{j+k=p} g_j \wedge h_k,\ p = 0, \dots, b+c) \\
&= \sum_q \left(\sum_{i+p=q} f_i \wedge \gamma_p\right) \\
&\quad = \left(\sum_i f_i\right) \wedge \left(\sum_p \gamma_p\right) \\
&\quad = \left(\sum_i f_i\right) \wedge \left(\sum_p \left(\sum_{j+k=p} g_j \wedge h_k\right)\right) \\
&\quad = \left(\sum_i f_i\right) \wedge \left(\left(\sum_j g_j\right) \wedge \left(\sum_k h_k\right)\right)
\end{aligned}
$$

Thus $\mathcal{A}^\bullet(V)$ is an associative algebra.

b)

$$\dim(\mathcal{A}^k(V)) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} & 0 \leq k \leq n \\ 0 & k > n \end{cases}$$

$$\implies \dim(\mathcal{A}^\bullet(V)) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = 2^n$$

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Problem 6

We need to identify the set $\Omega^k(M)$ of differential k-forms on M with the space of smooth functions that map M to the set of alternating k-tensors on $\mathbb{R}^d$.

As was discussed in the lectures, we can think of each element ω in $\Omega^k(M)$ as a map from M to $\mathcal{A}^k(T_pM)$, which maps a point on a manifold to an alternating k-tensor on the tangent space of the manifold point. Thus if we can show that the tangent space of a point in $M = \mathbb{R}^d$ is simply $\mathbb{R}^d$ itself, then we have shown what has been asked.

For any point $p \in M = \mathbb{R}^d$ we can consider the identity mapping $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ defined simply as $\alpha(p) = p$. This mapping satisfies all the necessary requirements to be a primitive coordinate patch for M. Thus, the tangent space at a point p is simply $T_pM = \text{im}(D\alpha(p)) = \text{im}(\mathbf{1}_{d\times d}) = \mathbb{R}^d$.

Thus any element of $\Omega^k(M)$ can be identified as a smooth function mapping M to $\mathcal{A}^k(\mathbb{R}^d)$.