MAU23206: Calculus on Manifolds
Homework 3 due 18/02/2022

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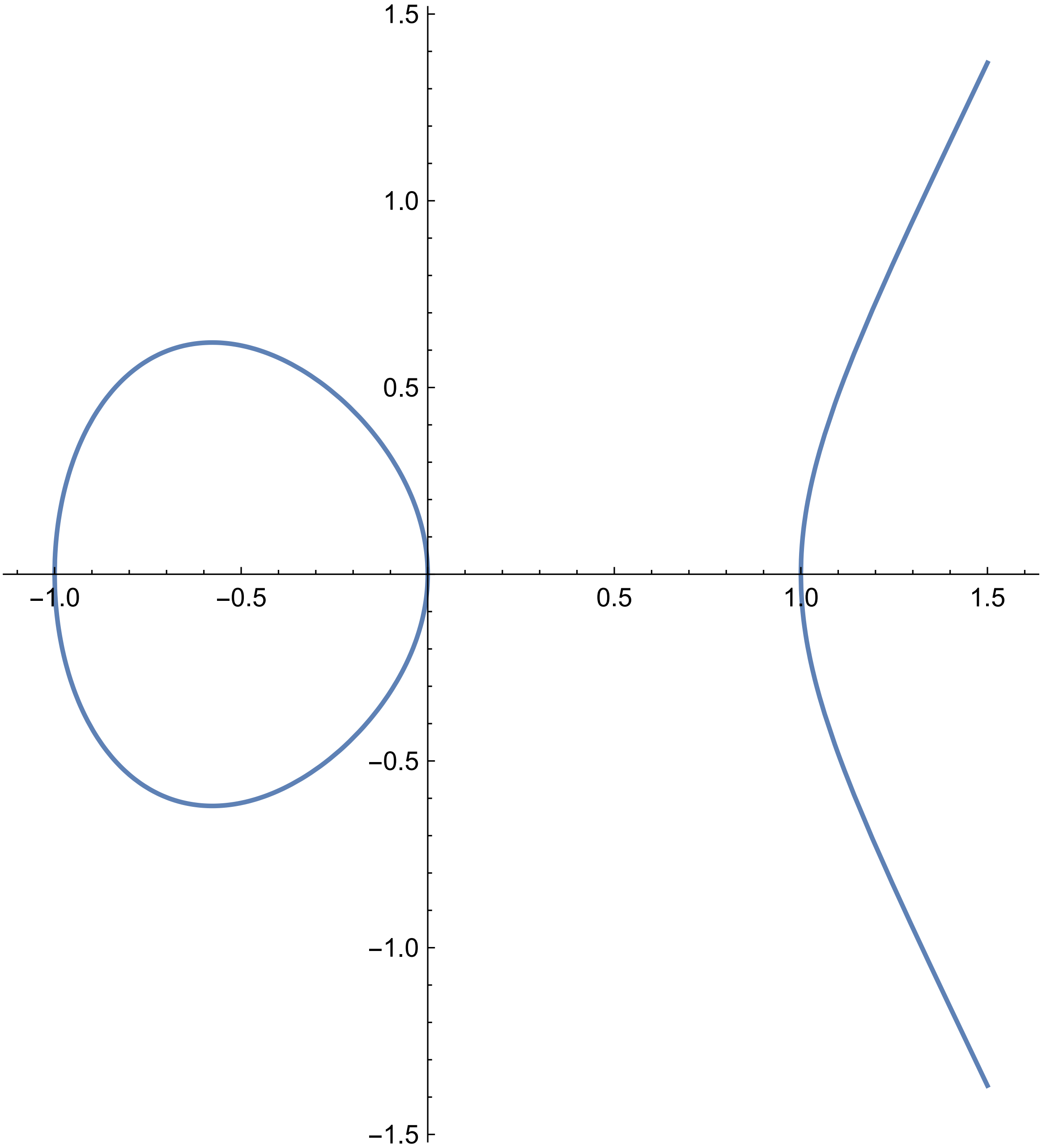
JS Theoretical Physics

Problem 1

a)

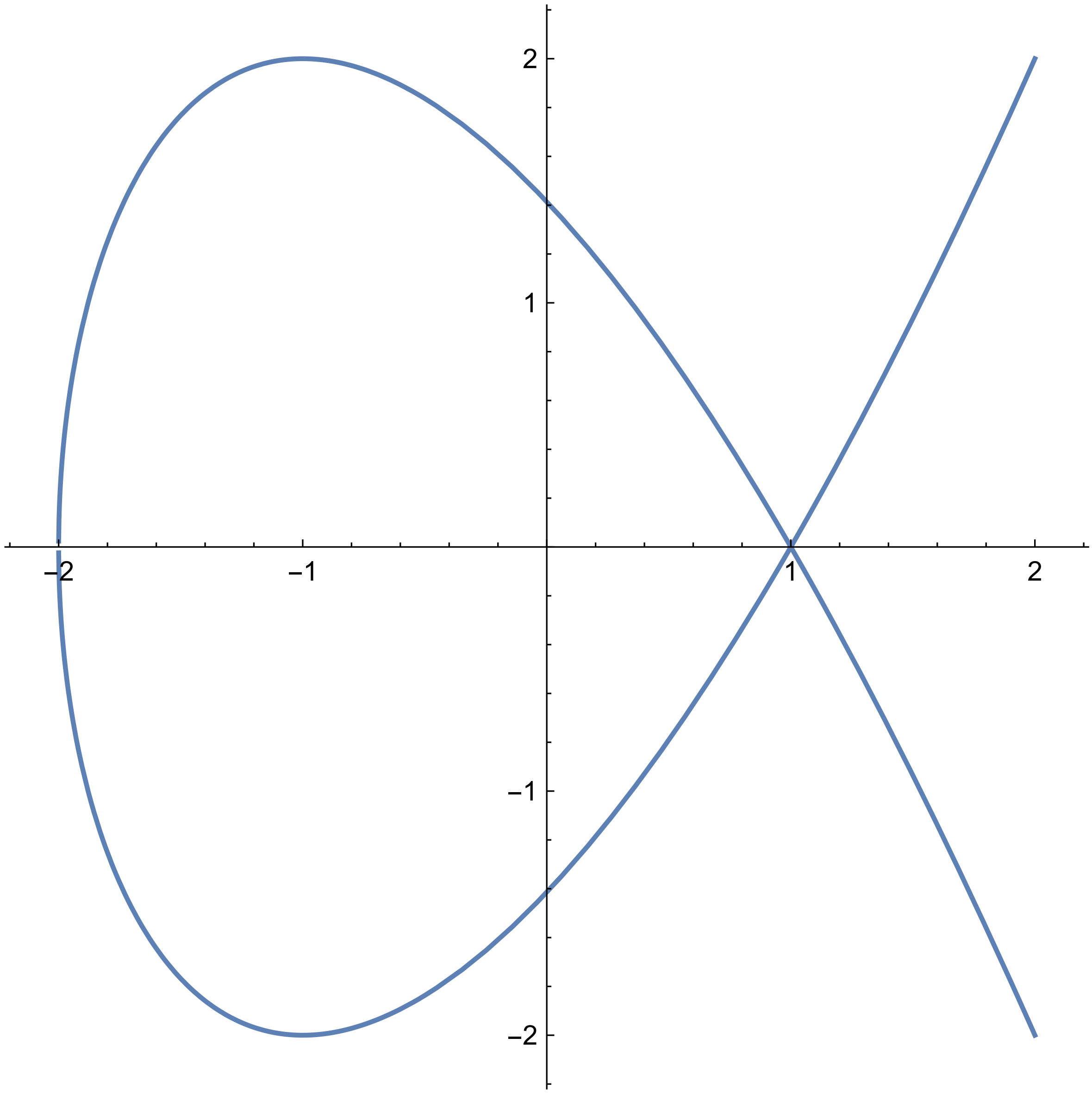
Below is the plot of $\left\{(x,y) \in \mathbb{R}^2 \,|\, y^2 = x^3 - x\right\}$ for $x \in [-1,2]$.



Since this is simply the disjoint union of two sets which are clearly manifolds, this set is a manifold.

b)

Below is the plot of $\{(x, y) \in \mathbb{R}^2 \,|\, y^2 = x^3 - 3x + 2\}$ for $x \in [0, 2]$.



From the plot, it is clear that there is a self-intersection at the point $(1, 0)$, as

$$\lim_{x\to 1^+} \sqrt{x^3 - 3x + 2} = \lim_{x\to 1^-} \sqrt{x^3 - 3x + 2} = \lim_{x\to 1^+} -\sqrt{x^3 - 3x + 2} = \lim_{x\to 1^-} -\sqrt{x^3 - 3x + 2} = \sqrt{(-1)^3 - 3(-1) + 2} = 0.$$

Excluding the point $(1, 0)$ from T would make it a manifold, as then everywhere else a close enough zoom of the set represents $\mathbb{R}^1$.

c)

We can assume that, for T to be self-intersecting, that y^2 must have a double root at some point p and a signle root at q.

$$\begin{aligned} x^3 + a\,x + b &= (x-p)^2\,(x-q) \\ &= x^3 + x^2\,(-q-2\,p) + x\,\left(2\,p\,q + p^2\right) - p^2\,q \end{aligned}$$

$$\begin{aligned} -q - 2\,p &= 0 && \implies q = -2\,p \\ a &= 2\,p\,q + p^2 && \implies a = -3\,p^2 \\ b &= -p^2\,q && \implies b = 2\,p^3 \end{aligned}$$

Thus if (a, b) are of the form $(-3\,p^2, 2\,p^3)$ then the graph of T self-intersects and thus is not a manifold, and so for T to be a manifold it cannot be in this form.

2.

Let $f(\vec{x}) \equiv r^2 - \left(R - \sqrt{x^2+y^2}\right)^2 - z^2$. Thus $\mathbb{T} = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, f(\vec{x}) = 0\right\}$. This set is simply the boundary of the "filled-in" torus

$$\mathfrak{T} = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 \leq r^2\right\} = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, f(\vec{x}) \geq 0\right\}.$$

Thus if we can show that $\mathfrak{T}$ is a manifold with boundary, then its boundary $\mathbb{T}$ must be a manifold without boundary.

$$\begin{aligned} Df &= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \\ &= \begin{pmatrix} 2\,x\left(\frac{R}{\sqrt{x^2+y^2-1}}\right) & 2\,y\left(\frac{R}{\sqrt{x^2+y^2-1}}\right) & -2\,z \end{pmatrix} \end{aligned}$$

$Df(\vec{x})$ only has rank 0 if $\vec{x} = \vec{0}$, however $\vec{0} \notin \mathbb{T}$ and so the rank of $Df(\vec{x})$ is always 1. Therefore $\mathfrak{T}$ is a manifold with boundary $\mathbb{T}$, and so $\mathbb{T}$ is a manifold without boundary.

3.

Let $p \in M$ be given, and a coordinate patch α that meets the required conditions such that $\alpha(x_0) = p$. Consider the subset of the geometric tangent space defined as

$$G_pM = \left\{ v \in \mathbb{R}^n \,\middle|\, \exists\gamma : \mathbb{R} \to M \text{ smooth} : v = \left.\frac{d\gamma}{dt}\right|_{t=0} \text{ and } \gamma(0) = p \right\}.$$

Pick a γ that meets the above conditions and consider the corresponding $v = \left.\frac{d\gamma}{dt}\right|_{t=0}$. We then have that $v \in \operatorname{im} D\alpha(x_0)$. For any arbitrary element $v' \in \operatorname{im} D\alpha(x_0)$ we can rotate and scale γ to obtain a new γ' that is still smooth and meets the relevant conditions to obtain the corresponding $v' = \left.\frac{d\gamma'}{dt}\right|_{t=0}$. Similarly, given a γ'' that meets the relevant conditions and the corresponding $v'' = \left.\frac{d\gamma''}{dt}\right|_{t=0}$, we can easily choose a vector w such that $D\alpha(x_0)w = v''$. Thus the spaces coincide.

The same cannot necessarily be said for a manifold with boundary, however. Given a boundary point of M and a γ that meets the relevant conditions, we will not be able to find a w as before, as $\operatorname{im} D\alpha(x_0)$ does not include vectors "away from" the boundary, yet a γ can provide these vectors.

4.

a)

Let $p \in M$ be given. Then we have a coordinate patch $\alpha : \mathbb{R}^d \to M$ that meets the required conditions such that $\alpha(x_0) = p$ for some $x_0 \in \mathbb{R}^d$. We then have that $D\alpha(x_0) : \mathbb{R}^d \to \mathbb{R}^n$ is smooth and has a continuous inverse whose derivative has rank d. Let $v \in T_pM$ be given such that $D\alpha(x_0)y_0 = v$ for some $y_0 \in \mathbb{R}^d$. Thus, $\beta : \mathbb{R}^{d+d} \to \mathbb{R}^{n+n}$ defined as $\beta(x, y) = (\alpha(x), D\alpha(x)y)$ for $x, y \in \mathbb{R}^d$ is also a coordinate patch for the point (p, v) as it is a composition of two functions that meet the coordinate patch requirements, and $\beta(x_0, y_0) = (p, v)$. Since $v \in T_pM \subset \mathbb{R}^n$ then β maps from $\mathbb{R}^{2d}$ to TM, and so TM is a manifold of dimension $2d$.

b)

$Df : TM \to TN$ is defined as $Df(p, v) = (f(p), Df(p)v)$. By definition, f is smooth, and so $f(p)$ and $Df(p)$ are both defined. Therefore, $f(p)$ and $Df(p)v$ are both well-defined, and so Df is a composition of smooth functions and is thus smooth.

c)

$$\begin{aligned}
(D(g \circ f))(p, v) &= ((g \circ f)(p), (D(g \circ f))(p)v) \\
&= (g(f(p)), Dg(f(p))Df(p)v) && \text{(by the chain rule)} \\
&= Dg(f(p), Df(p)v) \\
&= Dg(Df(p, v)) \\
&= (Dg \circ Df)(p, v)
\end{aligned}$$

5.

Statement 1. *Let $f : V^k \to \mathbb{R}$, $g : V^l \to \mathbb{R}$, and $h : V^m \to \mathbb{R}$ be tensors. Define the tensor product $\otimes$ and the $k + l$ tensor $f \otimes g : \mathbb{R}^{k+l} \to \mathbb{R}$ by*

$$(f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}).$$

1. $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.
2. $(\lambda f) \otimes g = \lambda (f \otimes g) = f \otimes (\lambda g)$.
3. $(f + g) \otimes h = f \otimes h + g \otimes h$,
$h \otimes (f + g) = h \otimes f + h \otimes g$.
4. $e^{(i_1,\ldots,i_n)} = e^{i_1} \otimes \ldots \otimes e^{i_n}$.

Proof. Denote $\mathbf{v}_i \in V^j = V \times \ldots \times V$, with $1 \leq i \leq j$.

1.

$$\begin{aligned}
(f \otimes (g \otimes h))(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m}) &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot (g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \cdot h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m})) \\
&= (f(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})) \cdot h(\mathbf{v}_{k+l+1}, \ldots, \mathbf{v}_{k+l+m}) \\
&= ((f \otimes g) \otimes h)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l+m})
\end{aligned}$$

2.

$$\begin{aligned}
((\lambda f) \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) &= (\lambda f(\mathbf{v}_1, \ldots, \mathbf{v}_k)) \cdot g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}) \\
&= \lambda (f(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})) \\
&= \lambda (f \otimes g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) \\
\text{also } &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot (\lambda g(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l})) \\
&= f \otimes (\lambda g)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l})
\end{aligned}$$

3. Let $k = l$.

$$\begin{aligned}(f+g)\otimes h(\mathbf{v}_1,\ldots,\mathbf{v}_{k+m}) &= (f+g)(\mathbf{v}_1,\ldots,\mathbf{v}_k)\cdot h(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+m})\\ &= (f(\mathbf{v}_1,\ldots,\mathbf{v}_k)+g(\mathbf{v}_1,\ldots,\mathbf{v}_k))\cdot h(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+m})\\ &= f(\mathbf{v}_1,\ldots,\mathbf{v}_k)\cdot h(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+m}) + g(\mathbf{v}_1,\ldots,\mathbf{v}_k)\cdot h(\mathbf{v}_{k+1},\ldots,\mathbf{v}_{k+m})\\ &= (f\otimes h)(\mathbf{v}_1,\ldots,\mathbf{v}_{k+m}) + (g\otimes h)(\mathbf{v}_1,\ldots,\mathbf{v}_{k+m})\\ &= (f\otimes h + g\otimes h)(\mathbf{v}_1,\ldots,\mathbf{v}_{k+m})\end{aligned}$$

$$\begin{aligned}h\otimes(f+g)(\mathbf{v}_1,\ldots,\mathbf{v}_{m+k}) &= h(\mathbf{v}_1,\ldots,\mathbf{v}_m)\cdot((f+g)(\mathbf{v}_{m+1},\ldots,\mathbf{v}_{m+k}))\\ &= h(\mathbf{v}_1,\ldots,\mathbf{v}_m)\cdot(f(\mathbf{v}_{m+1},\ldots,\mathbf{v}_{m+k})+g(\mathbf{v}_{m+1},\ldots,\mathbf{v}_{m+k}))\\ &= h(\mathbf{v}_1,\ldots,\mathbf{v}_m)\cdot f(\mathbf{v}_{m+1},\ldots,\mathbf{v}_{m+k}) + h(\mathbf{v}_1,\ldots,\mathbf{v}_m)\cdot g(\mathbf{v}_{m+1},\ldots,\mathbf{v}_{m+k})\\ &= (h\otimes f)(\mathbf{v}_1,\ldots,\mathbf{v}_{m+k}) + (h\otimes g)(\mathbf{v}_1,\ldots,\mathbf{v}_{m+k})\\ &= (h\otimes f + h\otimes g)(\mathbf{v}_1,\ldots,\mathbf{v}_{m+k})\end{aligned}$$

4.

$$\begin{aligned}e^{(i_1,\ldots,i_n)}(\mathbf{v}_1,\ldots,\mathbf{v}_n) &= e^{i_1}(\mathbf{v}_1)\cdot\ldots\cdot e^{i_n}(\mathbf{v}_n)\\ &= (e^{i_1}\otimes\ldots\otimes e^{i_n})(\mathbf{v}_1,\ldots,\mathbf{v}_n)\end{aligned}$$

□

Statement 2. *Define the averaging operator $A : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ by*

$$Af = \sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\, f^\sigma,$$

where f is a k-tensor on V.

1. *A is linear.*
2. *$Af \in A^k(V)$.*
3. *$f \in A^k(V) \implies Af = k!\, f$.*

Proof. 1. $f \to f^\sigma$ is linear, and so A must also be linear as it is a linear combination of linear functions.

2.

$$\begin{aligned}(Af)^\tau &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,(f^\sigma)^\tau\\ &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,(\operatorname{sgn}\tau)^2\, f^{\tau\sigma}\\ &= (\operatorname{sgn}\tau)\sum_{\sigma\in S_k}(\operatorname{sgn}\tau\sigma)\, f^{\tau\sigma}\\ &= (\operatorname{sgn}\tau)\, Af\end{aligned}$$

3. Let f be an alternating tensor.

$$\begin{aligned}Af &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\, f^\sigma\\ &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,(\operatorname{sgn}\sigma)\, f\\ &= \sum_{\sigma\in S_k} f\\ &= k!\, f\end{aligned}$$

□

Statement 3. *Let $A : V \to W$ be a linear map, and $f \in \mathcal{L}^k(W)$. Define the dual transformation $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ by*

$$(A^* f)(\mathbf{v}_1, \dots, \mathbf{v}_k) = f(A\mathbf{v}_1, \dots, A\mathbf{v}_k).$$

1. *A^* is linear.*
2. *$A^*(f \otimes h) = A^* f \otimes A^* h$ for $h \in \mathcal{L}^l(W)$.*
3. *$(AB)^* = B^* A^*$ for a linear map $B : U \to V$.*

Proof. 1. Let $g \in \mathcal{L}^k(W)$.

$$\begin{aligned}(A^*(a\,f + b\,g))(\mathbf{v}_1, \dots, \mathbf{v}_k) &= (a\,f + b\,g)(A\mathbf{v}_1, \dots, A\mathbf{v}_k)\\ &= a\,f(A\mathbf{v}_1, \dots, A\mathbf{v}_k) + b\,g(A\mathbf{v}_1, \dots, A\mathbf{v}_k)\\ &= a\,A^* f(T(\mathbf{v}_1), \dots, A(\mathbf{v}_k)) + b\,A^* g(T(\mathbf{v}_1), \dots, A(\mathbf{v}_k))\\ &= (a\,A^* f + b\,A^* g)(\mathbf{v}_1, \dots, \mathbf{v}_k)\end{aligned}$$

2.

$$\begin{aligned}A^*(f \otimes h)(\mathbf{v}, \dots, \mathbf{v}_{k+l}) &= (f \otimes h)(A\mathbf{v}_1, \dots, A\mathbf{v}_{k+l})\\ &= f(A\mathbf{v}_1, \dots, A\mathbf{v}_k) \cdot h(A\mathbf{v}_{k+1}, \dots, A\mathbf{v}_{k+l})\\ &= A^* f(\mathbf{v}, \dots, \mathbf{v}_k) \cdot A^* h(A\mathbf{v}_{k+1}, \dots, A\mathbf{v}_{k+l})\\ &= (A^* f \otimes A^* h)(\mathbf{v}, \dots, \mathbf{v}_{k+l})\end{aligned}$$

3.

$$\begin{aligned}(AB)^* f(\mathbf{v}_1, \dots, \mathbf{v}_k) &= f(AB\mathbf{v}_1, \dots, AB\mathbf{v}_k)\\ &= A^* f(B\mathbf{v}_1, \dots, B\mathbf{v}_k)\\ &= B^* A^* f(\mathbf{v}_1, \dots, \mathbf{v}_k)\end{aligned}$$

□