MAU23206: Calculus on Manifolds
Homework 2 due 11/02/2022

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Problem 1

(a)

$$\begin{aligned}\alpha_1(s) &= \left(\frac{2\,s}{s^2+1}, \frac{s^2-1}{s^2+1}\right)\\ &= (x,y)\end{aligned}$$

$$\begin{aligned}\frac{x}{s}+y &= \frac{2+s^2-1}{s^2+1}\\ &= 1\\ \implies s &= \frac{x}{1-y}\end{aligned}$$

$$\alpha_1^{-1}((x,y)) = \frac{x}{1-y}$$

$$\begin{aligned}\alpha_2(s) &= \left(\frac{2\,s}{s^2+1}, \frac{1-s^2}{s^2+1}\right)\\ &= (x,y)\end{aligned}$$

$$\begin{aligned}&\frac{x}{s}-y\\ &= 1\\ \implies s &= \frac{x}{1+y}\end{aligned}$$

$$\alpha_2^{-1}((x,y)) = \frac{x}{1+y}$$

(b)

$$\begin{aligned}\alpha_2^{-1}\circ\alpha_1(s) &= \alpha_2^{-1}\left(\frac{2\,s}{s^2+1}, \frac{s^2-1}{s^2+1}\right)\\ &= \frac{\frac{2\,s}{s^2+1}}{1+\frac{s^2-1}{s^2+1}}\\ &= \frac{2\,s}{s^2+1}\,\frac{s^2+1}{2\,s^2}\\ \alpha_2^{-1}\circ\alpha_1(s) &= \frac{1}{s}\end{aligned}$$

(c)

Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as $f(\vec{x}) = 1-||\vec{x}||$. Consider $\vec{p}\in S^1$. Then $f(\vec{p}) = 0$. Since $Df(\vec{x}) = \begin{pmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2}\end{pmatrix} = \begin{pmatrix}-2\,x_1 & -2\,x_2\end{pmatrix}$ and $\vec{p}\neq\vec{0}$ then $Df(\vec{p})$ has rank 1. Then there must exist an index i such that $\frac{\partial f}{\partial x_i}(\vec{p})\neq 0$. We can define two functions $F_1, F_2:\mathbb{R}^2\to\mathbb{R}^2$ defined as $F_1((x_1,x_2)) = (f(\vec{x}), x_2)$ and $F_2((x_1,x_2)) = (x_1, f(\vec{x}))$. The derivatives at $\vec{p}$ of these functions are given by $DF_1(\vec{p}) = \begin{pmatrix}\frac{\partial f}{\partial x_1}(\vec{p}) & 0\\ \frac{\partial f}{\partial x_2}(\vec{p}) & 1\end{pmatrix}$ and $DF_2(\vec{p}) = \begin{pmatrix}1 & \frac{\partial f}{\partial x_1}(\vec{p})\\ 0 & \frac{\partial f}{\partial x_2}(\vec{p})\end{pmatrix}$, which are non singular for $p_1\neq 0$ and $p_2\neq 0$, respectively. Then by the inverse function theorem there exists an open neighbourhood $A_{1,2}$ of $\vec{p}$ in $\mathbb{R}^n$ and a set $B_{1,2}$ open in $\mathbb{R}^n$ such that $F_{1,2}|_{A_{1,2}}: A_{1,2}\to B_{1,2}$ is a diffeomorphism. Setting $\alpha_{1,2} = F_{1,2}^{-1}|_{A_{1,2}}$ gives the required coordinate patches for $\vec{p}$, with α_1 covering all points where $p_1\neq 0$ and α_2 covering all points where $p_2\neq 0$.

Problem 2

(a)

Consider $U = [0, 1)$, $V = [0, 1)$, and $\alpha : U \to V$ defined as $\alpha(x) = x$. Clearly, U is open in $\mathbb{H}^1$, V is open in I, α is smooth, α and α^{-1} are continuous, and $D\alpha(x) = 1$ for all $x \in U$.

Now consider $U' = [0, 1)$, $V' = (0, 1]$, and $\alpha' : U' \to V'$ defined as $\alpha'(x) = 1 - x$. Clearly, U' is open in $\mathbb{H}^1$, V' is open in I, α' is smooth, α' and α'^{-1} are continuous, and $D\alpha'(x) = -1$ for all $x \in U$.

Thus I is a 1-manifold in $\mathbb{R}^1$.

(b)

If $I \times I$ was a 2-manifold in $\mathbb{R}^2$ then its boundary must be a 1-manifold without boundary. However, the boundary of $I \times I$ is similar to I itself, which is a manifold with boundary. Thus, $I \times I$ cannot be a 2-manifold.

Problem 3

It is clear that $K_i \subset K_{i+1}$, since $B_i(\vec{0}) \subset \overline{B_{i+1}(\vec{0})}$ and $B_{2^{-i}}(\vec{x}) \supset B_{2^{-i-1}}(\vec{x})$ for any $\vec{x} \in A^c$. Let $k \in \partial(K_{i+1})$. If $k \in \partial\Big(\overline{B_{i+1}(\vec{0})}\Big)$ then $k \notin \overline{B_i(\vec{0})} \supset K_i$. If $k \in \partial\Big(\overline{B_{2^{-i-1}}(\vec{x})}\Big)$ for some $\vec{x} \in A^c$ then $k \in B_{2^{-i}} \not\subset K_i$. Therefore no boundary point of K_{i+1} is also a point of K_i, and so $K_i \subset \operatorname{int}(K_{i+1})$.

Problem 4

Consider the composition $\delta \circ \gamma \circ \beta \circ \alpha$, where the functions are defined as

$$\alpha(\vec{x}) = ||\vec{x}||^2 = \sum_{i=1}^{n} x_i^2 \qquad \beta(x) = x - 1 \qquad \gamma(x) = \frac{1}{x} \qquad \delta(x) = e^x.$$

Then we can write φ as

$$\varphi(\vec{x}) = \begin{cases} \delta(\gamma(\beta(\alpha(\vec{x})))) & \text{if } ||\vec{x}|| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since each of the functions α, β, γ, δ are smooth on the relevant domains then their composition is also smooth by the chain rule. Thus φ is infinitely differentiable at all points where $||\vec{x}|| \neq 1$. However, since φ and its derivatives tend to 0 as $||\vec{x}||$ tends to 1, then φ is also infinitely differentiable at $||\vec{x}|| = 1$, and thus φ is smooth.

Problem 5

Let $\varepsilon > 0$ be given. Consider the sets $U(\vec{x}) = \{\vec{y} \in A \mid ||f(\vec{y}) - f(\vec{x})|| < \varepsilon\}$ for points $\vec{x} \in A$. The union of these sets for all $\vec{x} \in A$ is equal to A and so we can consider a partition of unity with $\{\phi_i\}$ where $S_i \subseteq U(\vec{x}_i)$ for some $\vec{x}_i \in U(\vec{x})$. Let $g(\vec{x}) = \sum_i \phi_i(\vec{x})\, f(\vec{x}_i)$, which is clearly smooth as it is a linear combination of smooth functions. We then have

$$\begin{aligned} ||f(\vec{x}) - g(\vec{x})|| &= \left|\left| f(\vec{x}) \sum_i \phi_i(\vec{x}) - \sum_i \phi_i(\vec{x})\, f(\vec{x}_i) \right|\right| \\ &\leq \sum_i \phi_i(\vec{x})\, ||f(\vec{x}_i) - f(\vec{x})|| \\ &< \sum_i \phi_i(\vec{x})\, \varepsilon \\ &= \varepsilon, \end{aligned}$$

as required.