MAU23206: Calculus on Manifolds
Homework 1 due 04/02/2022

Ruaidhrí Campion
19333850
JS Theoretical Physics

Problem 1

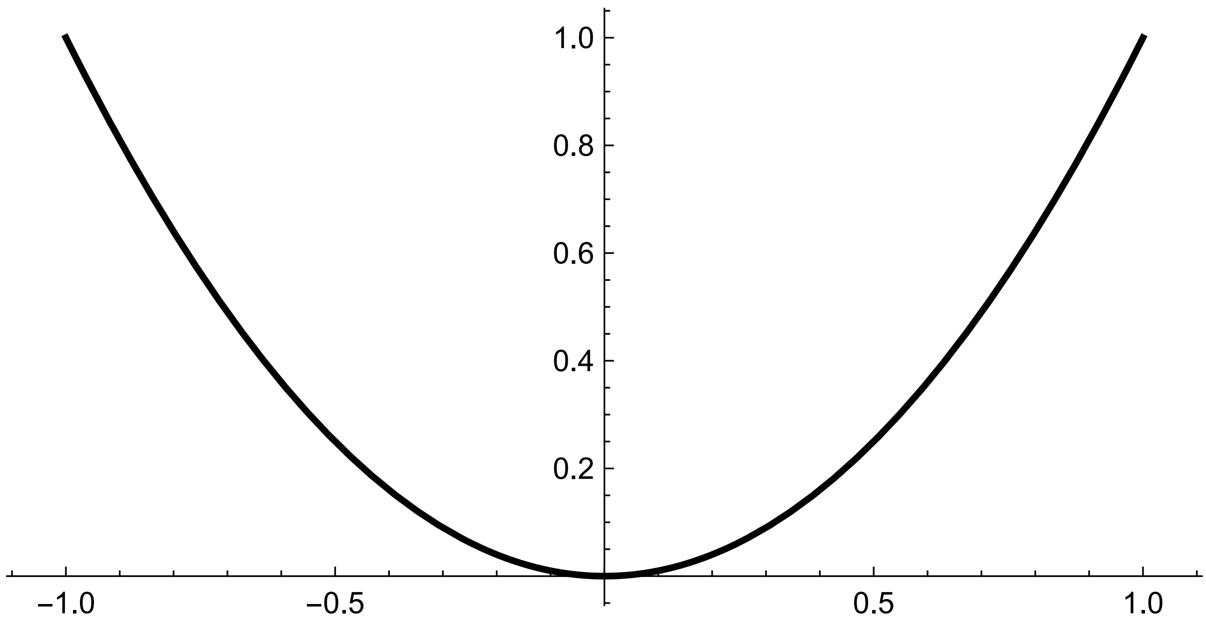
Consider the points $x \in U$ and $y \in V$ such that $y = f(x)$. Since f is bijective, we have that $f\big(f^{-1}(y)\big) = y$. From the chain rule, we have $Df\big(f^{-1}(y)\big) \circ Df^{-1}(y) = I$. Since $y = f(x)$ and $f^{-1}(y) = x$, this simplifies to $Df(x) \circ Df^{-1}(x) = I$. Thus $Df(x)$ is an invertible matrix with inverse $Df^{-1}(x)$, and so $Df(x)$ must be a square matrix. Since f is bijective then this is true for all points $x \in U$, and so $Df(x)$ is a square matrix for all points $x \in U$, i.e. $n = m$.

Problem 2

It is obvious that the mapping $\alpha : \mathbb{R} \to M$ defined as $\alpha(t) = \big(t^3, t^2\big)$ would not be a coordinate patch covering M if M were a manifold, as $D\alpha(0) = (0, 0)$, which does not have rank 1 as required. However, given any other mapping $\beta : \mathbb{R} \to M$, we can decompose it into a composition of continuous mappings $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined as $\gamma(t) = \alpha(t)$ and some $\delta : \mathbb{R}^2 \to M$, where $\beta = \delta \circ \gamma$. Thus by the chain rule, $D\beta(0) = D\delta(\gamma(0)) \circ D\gamma(0) = D\delta((0, 0)) \circ (0, 0) = (0, 0)$, which will also not have rank 1. Therefore M cannot be a manifold.

Problem 3

(a)



(b)

The map α is infinitely differentiable ($D\alpha(x) = (1, 2x)$, $D^2\alpha(x) = (0, 2)$, $D^n\alpha(x) = (0, 0)\,\forall\, n \geq 3$) and thus is of class C^∞. α is clearly a bijection, since $\alpha(x)$ and $\alpha^{-1}(x, x^2)$ each have single outputs, namely (x, x^2) and x, respectively. The inverse is defined as $\alpha^{-1}(x, y) = x$, which is continuous since it simply projects a vector onto its first component, and so α has a continuous inverse. $D\alpha(x) = (1, 2x)$ has rank 1 for any value of $x \in \mathbb{R}$. Therefore M is a 1-manifold in $\mathbb{R}$ of class C^∞, covered by the single coordinate patch α.

Problem 4

Define $\alpha : U \to \Gamma_f$ as the map $\alpha(x) = (x, f(x)) = (x_1, \ldots, x_n, f_1(x), \ldots, f_m(x))$. Since f is smooth then the partial derivatives of α exist and are continuous, and so the derivative matrices are given by

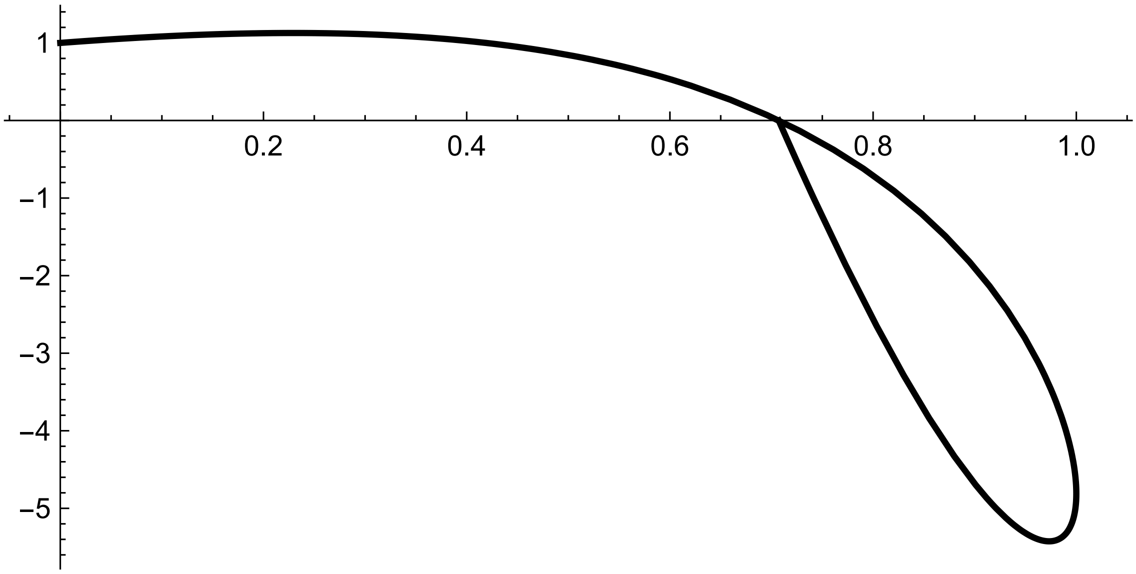
$$\begin{aligned}
D\alpha &= J \\
&= \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial x_1} & \cdots & \dfrac{\partial \alpha_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \alpha_{m+n}}{\partial x_1} & \cdots & \dfrac{\partial \alpha_{m+n}}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}
\end{aligned}$$

$$D^l\alpha = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \dfrac{\partial^l f_1}{\partial x_1^l} & \cdots & \dfrac{\partial^l f_1}{\partial x_n^l} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^l f_m}{\partial x_1^l} & \cdots & \dfrac{\partial^l f_m}{\partial x_n^l} \end{pmatrix} \forall\, l \geq 2.$$

α is infinitely differentiable and thus is smooth. α is clearly a bijection, since $\alpha(x)$ and $\alpha^{-1}(x, f(x))$ each have single outputs, namely $(x, f(x))$ and x, respectively. The inverse is defined as $\alpha^{-1}(x, y) = x$ (where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$), which is continuous since it simply a projects a vector onto its first component, and so α has a continuous inverse. Since the first n rows and columns are simply the $n \times n$ identity matrix and the partial derivatives of f are defined, the derivative matrix $D\alpha$ has rank n for any $x \in \mathbb{R}^n$. Therefore Γ_f is a smooth n-dimensional manifold, covered by the coordinate patch $\alpha(x) = (x, f(x))$.

Problem 5

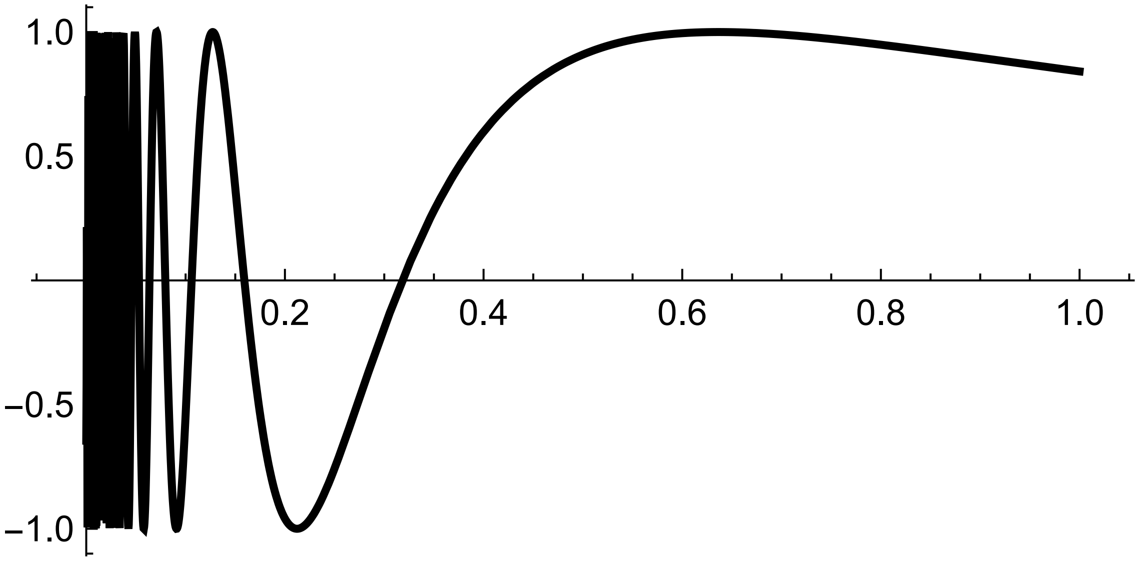
- The curve S is given below.



For $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$, the map defined in S returns $\left(\frac{1}{\sqrt{2}}, 0\right)$, and the graph shows that the curve intersects itself. Thus, no neighbourhood of this point will resemble $\mathbb{R}$, and so S is not a manifold.

- Consider the map $\alpha : (0, 1] \to A$, where A is the graph in question, defined as $\alpha(x) = \left(x, \sin\left(\frac{1}{x}\right)\right)$. This map is infinitely differentiable for all $x \in (0, 1]$ (only undefined for $x = 0$) and is clearly a bijection with a continuous inverse (projection mapping). Its derivative is given by $D\alpha(x) = \left(1, -\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right)$ which has rank 1 for any $x \neq 0 \notin (0, 1]$. Thus the graph in question is a 1-manifold.

- The subset is given below.



The origin (0,0) is on the subset as it is on the line segment $\left\{(0, y) \in \mathbb{R}^2 \,|\, y \in [-1, 1]\right\}$. Any neighbourhood about the origin will contain an infinite number of lines from the part of the subset given by A above, thus not resembling $\mathbb{R}$, and so the subset is not a manifold.