MAU34401: Classical Field Theory
Homework 4 due 06/12/2021

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JS Theoretical Physics

1.

(i)

$$\begin{aligned}
\mathcal{L}_\phi(\phi_1', \phi_2') &= \frac{1}{2}\,\partial_\mu\phi_1'\,\partial^\mu\phi_1' + \frac{1}{2}\,\partial_\mu\phi_2'\,\partial^\mu\phi_2' - V\Big((\phi_1')^2 + (\phi_2')^2\Big) \\
&= \frac{1}{2}\,\partial_\mu(\phi_1\cos\alpha + \phi_2\sin\alpha)\,\partial^\mu(\phi_1\cos\alpha + \phi_2\sin\alpha) \\
&\quad + \frac{1}{2}\,\partial_\mu(-\phi_1\sin\alpha + \phi_2\cos\alpha)\,\partial^\mu(-\phi_1\sin\alpha + \phi_2\cos\alpha) \\
&\quad - V\left(\phi_1^2\cos^2\alpha + \phi_2^2\sin^2\alpha + 2\,\phi_1\,\phi_2\cos\alpha\sin\alpha + \phi_1^2\sin^2\alpha + \phi_2^2\cos^2\alpha - 2\,\phi_1\,\phi_2\cos\alpha\sin\alpha\right) \\
&= \frac{1}{2}\left(\cos^2\alpha\,\partial_\mu\phi_1\,\partial^\mu\phi_1 + \cos\alpha\sin\alpha\,(\partial_\mu\phi_1\,\partial^\mu\phi_2 + \partial_\mu\phi_2\,\partial^\mu\phi_1) + \sin^2\alpha\,\partial_\mu\phi_2\,\partial^\mu\phi_2\right) \\
&\quad + \frac{1}{2}\left(\sin^2\alpha\,\partial_\mu\phi_1\,\partial^\mu\phi_1 - \cos\alpha\sin\alpha\,(\partial_\mu\phi_1\,\partial^\mu\phi_2 + \partial_\mu\phi_2\,\partial^\mu\phi_1) + \cos^2\alpha\,\partial_\mu\phi_2\,\partial^\mu\phi_2\right) \\
&\quad - V\left(\phi_1^2 + \phi_2^2\right) \\
&= \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - V\left(\phi_1^2 + \phi_2^2\right) \\
&= \mathcal{L}_\phi(\phi_1, \phi_2)
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{L}_\phi &= \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - V\left(\phi_1^2 + \phi_2^2\right) \\
&= \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi_1\,\partial_\nu\phi_1 + \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi_2\,\partial_\nu\phi_2 - V\left(\phi_1^2 + \phi_2^2\right)
\end{aligned}$$

(re-writing $\mathcal{L}_\phi$ in terms of the metric tensor for convenience)

$$\partial_\lambda\left(\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\lambda\phi_i)}\right) = \frac{\partial\mathcal{L}_\phi}{\partial\phi_i} \qquad \text{(E-L equations of motion)}$$

$$\begin{aligned}
\partial_\lambda\left(\frac{1}{2}\,g^{\mu\nu}\,\delta^\lambda{}_\mu\,\partial_\nu\phi_i + \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi_i\,\delta^\lambda{}_\nu\right) &= -\frac{\partial V\left(\phi_1^2 + \phi_2^2\right)}{\partial\phi_i} \\
\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\partial_\nu\phi_i + \frac{1}{2}\,g^{\mu\nu}\,\partial_\nu\partial_\mu\phi_i &= -\frac{\partial V\left(\phi_1^2 + \phi_2^2\right)}{\partial(\phi_1^2 + \phi_2^2)}\,\frac{\partial\left(\phi_1^2 + \phi_2^2\right)}{\partial\phi_i} \\
\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\partial_\nu\phi_i + \frac{1}{2}\,g^{\nu\mu}\,\partial_\nu\partial_\mu\phi_i &= -V'\cdot 2\,\phi_i \\
\partial_\mu\partial^\mu\phi_i &= -2\,\phi_i\,V' \qquad \text{(simplified equations of motion)}
\end{aligned}$$

$$\begin{aligned}
\partial_\mu\mathcal{J}^\mu &= \partial_\mu(\phi_2\,\partial^\mu\phi_1 - \phi_1\,\partial^\mu\phi_2) \\
&= \partial_\mu\phi_2\,\partial^\mu\phi_1 + \phi_2\,\partial_\mu\partial^\mu\phi_1 - \partial_\mu\phi_1\,\partial^\mu\phi_2 - \phi_1\,\partial_\mu\partial^\mu\phi_2 \\
&= g^{\mu\nu}\,\partial_\mu\phi_2\,\partial_\nu\phi_1 - g^{\mu\nu}\,\partial_\mu\phi_1\,\partial_\nu\phi_2 + 2\,\phi_1\,\phi_2\,V' - 2\,\phi_2\,\phi_1\,V' \\
&= g^{\mu\nu}\,\partial_\mu\phi_2\,\partial_\nu\phi_1 - g^{\nu\mu}\,\partial_\nu\phi_2\,\partial_\mu\phi_1 + 0 \\
&= 0
\end{aligned}$$

(iii)

$$\begin{aligned}
\partial_\mu\psi^*\,\partial^\mu\psi - V\Big(2\,|\psi|^2\Big) &= \partial_\mu\left(\frac{\phi_1 - i\,\phi_2}{\sqrt{2}}\right)\partial^\mu\left(\frac{\phi_1 + i\,\phi_2}{\sqrt{2}}\right) - V(2\,\psi\,\psi^*)\\
&= \frac{1}{\sqrt{2}}\left(\partial_\mu\phi_1 - i\,\partial_\mu\phi_2\right)\frac{1}{\sqrt{2}}\left(\partial^\mu\phi_1 + i\,\partial^\mu\phi_2\right) - V\left(2\left(\frac{\phi_1 + i\,\phi_2}{\sqrt{2}}\right)\left(\frac{\phi_1 - i\,\phi_2}{\sqrt{2}}\right)\right)\\
&= \frac{1}{2}\left(\partial_\mu\phi_1\,\partial^\mu\phi_1 + i\left(\partial_\mu\phi_1\,\partial^\mu\phi_2 - \partial_\mu\phi_2\,\partial^\mu\phi_1\right) + \partial_\mu\phi_2\,\partial^\mu\phi_2\right) - V\left(2\,\frac{\phi_1^2 + \phi_2^2}{2}\right)\\
&= \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - V\left(\phi_1^2 + \phi_2^2\right)\\
&= \mathcal{L}_\phi
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\phi &= \partial_\mu\psi^*\,\partial^\mu\psi - V\Big(2\,|\psi|^2\Big)\\
&= g^{\mu\nu}\,\partial_\mu\psi^*\,\partial_\nu\psi - V(2\,\psi\,\psi^*) \qquad \text{(again re-writing } \mathcal{L}_\phi \text{ for convenience)}
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda\left(\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\lambda\psi_i)}\right) &= \frac{\partial\mathcal{L}_\phi}{\partial\psi_i} \qquad \text{(where } \psi_1 = \psi,\ \psi_2 = \psi^*\text{)}\\
\partial_\lambda\left(g^{\mu\nu}\,\frac{\partial(\partial_\mu\psi^*\,\partial_\nu\psi)}{\partial(\partial_\lambda\psi_i)}\right) &= -\frac{\partial V(2\,\psi\,\psi^*)}{\partial(2\,\psi\,\psi^*)}\,\frac{\partial(2\,\psi\,\psi^*)}{\partial\psi_i}\\
g^{\mu\nu}\,\partial_\lambda\left(\delta^\lambda{}_\mu\,\delta^i{}_2\,\partial_\nu\psi + \partial_\mu\psi^*\,\delta^\lambda{}_\nu\,\delta^i{}_1\right) &= -V'\cdot 2\left(\delta^i{}_1\,\psi^* + \psi\,\delta^i{}_2\right)\\
g^{\mu\nu}\delta^i{}_2\,\partial_\mu\,\partial_\nu\psi + g^{\mu\nu}\,\delta^i{}_1\partial_\nu\,\partial_\mu\psi^* &= -2\left(\delta^i{}_1\,\psi^* + \psi\,\delta^i{}_2\right)V'\\
g^{\mu\nu}\delta^i{}_2\,\partial_\mu\,\partial_\nu\psi + g^{\nu\mu}\,\delta^i{}_1\partial_\nu\,\partial_\mu\psi^* &= -2\left(\delta^i{}_1\,\psi^* + \psi\,\delta^i{}_2\right)V'\\
g^{\mu\nu}\,\partial_\mu\partial_\nu\psi_j &= -2\,\psi_j\,V' \qquad (i \neq j, \text{ i.e. } i = 1,2 \iff j = 2,1)\\
\partial_\mu\partial^\mu\psi_j &= -2\,\psi_j\,V'
\end{aligned}$$

$$\partial_\mu\partial^\mu\psi = -2\,\psi\,V' \qquad\qquad \partial_\mu\partial^\mu\psi^* = -2\,\psi^*\,V'$$

If the real and imaginary parts of the above expressions are considered separately, then the resulting expressions are simply the equations of motion in terms of ϕ_1 and ϕ_2.

2.

(a)

$$\begin{aligned}
\mathcal{L}_{\text{CS}} \to \mathcal{L}'_{\text{CS}} &= \epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta(A_\gamma + \partial_\gamma\Lambda) \\
&= \epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,(\partial_\beta A_\gamma + \partial_\beta\partial_\gamma\Lambda) \\
&= \epsilon^{\alpha\beta\gamma}\,(A_\alpha\,\partial_\beta A_\gamma + A_\alpha\,\partial_\beta\partial_\gamma\Lambda + \partial_\alpha\Lambda\,\partial_\beta A_\gamma + \partial_\alpha\Lambda\,\partial_\beta\partial_\gamma\Lambda) \\
&= \mathcal{L}_{\text{CS}} + \epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda + \epsilon^{\alpha\beta\gamma}\,\partial_\alpha\Lambda\,\partial_\beta A_\gamma
\end{aligned}$$

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda &= \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda + \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda \\
&= \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda + \frac{1}{2}\,\epsilon^{\alpha\gamma\beta}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\gamma\partial_\beta\Lambda \\
&= \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\beta\partial_\gamma\Lambda - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,\partial_\gamma\partial_\beta\Lambda \\
&= \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,(A_\alpha + \partial_\alpha\Lambda)\,(\partial_\beta\partial_\gamma\Lambda - \partial_\gamma\partial_\beta\Lambda) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma}\,\partial_\alpha\Lambda\,\partial_\beta A_\gamma &= \epsilon^{\alpha\beta\gamma}\,\partial_\alpha\Lambda\,\partial_\beta A_\gamma + \epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma - \epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma \\
&= \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big) - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma \\
&= \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big) - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma - \frac{1}{2}\,\epsilon^{\beta\alpha\gamma}\,\Lambda\,\partial_\beta\partial_\alpha A_\gamma \\
&= \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big) - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\alpha\partial_\beta A_\gamma + \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,\partial_\beta\partial_\alpha A_\gamma \\
&= \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big) - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\Lambda\,(\partial_\alpha\partial_\beta A_\gamma - \partial_\beta\partial_\alpha A_\gamma) \\
&= \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big)
\end{aligned}$$

$$\implies \mathcal{L}_{\text{CS}} \to \mathcal{L}_{\text{CS}} + \partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big)$$

The difference in the transformed Lagrangian and the original Lagrangian is not necessarily zero, and so the Lagrangian is not invariant under this transformation. The effect on the action can be found by considering only the added term, i.e.

$$\begin{aligned}
\delta S &= S - S' \\
&= \int d^3x\,(\mathcal{L}_{\text{CS}} - \mathcal{L}'_{\text{CS}}) \\
&= \int d^3x\,\partial_\alpha\big(\epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\big) \\
&= \sum_\alpha \epsilon^{\alpha\beta\gamma}\Lambda\,\partial_\beta A_\gamma\bigg|_{-\infty}^{\infty}
\end{aligned}$$

If the surface terms fall off at infinity fast enough, then there is no effect on the action, i.e. $\delta S = 0$.

(b)

$$
\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}_{\text{CS}}}{\partial(\partial_\mu A_i)} \right) &= \frac{\partial \mathcal{L}_{\text{CS}}}{\partial A_i} \\
\partial_\mu \left(\epsilon^{\alpha\beta\gamma} \, A_\alpha \, \delta^\mu{}_\beta \, \delta^i{}_\gamma \right) &= \epsilon^{\alpha\beta\gamma} \, \delta^i{}_\alpha \, \partial_\beta A_\gamma - \delta^i{}_\alpha \, \mathcal{J}^\alpha \\
\epsilon^{\alpha\beta i} \, \partial_\beta A_\alpha &= \epsilon^{i\beta\gamma} \, \partial_\beta A_\gamma - \mathcal{J}^i \\
\mathcal{J}^i &= \epsilon^{i\beta\gamma} \, \partial_\beta A_\gamma - \epsilon^{\alpha\beta i} \, \partial_\beta A_\alpha \\
&= \epsilon^{i\beta\gamma} \, (\partial_\beta A_\gamma - \partial_\gamma A_\beta) \\
\mathcal{J}^i &= \epsilon^{i\beta\gamma} \, F_{\beta\gamma}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathcal{J}^\alpha &= \epsilon^{\alpha\beta\gamma} \, F_{\beta\gamma} \\
\partial_\alpha \mathcal{J}^\alpha &= \epsilon^{\alpha\beta\gamma} \, \partial_\alpha F_{\beta\gamma} \\
&= \epsilon^{\alpha\beta\gamma} \, (\partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta) \\
&= \epsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta A_\gamma - \epsilon^{\alpha\beta\gamma} \, \partial_\alpha \partial_\gamma A_\beta \\
&= \epsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta A_\gamma - \epsilon^{\beta\gamma\alpha} \, \partial_\beta \partial_\alpha A_\gamma \\
&= \epsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta A_\gamma - \epsilon^{\alpha\beta\gamma} \, \partial_\beta \partial_\alpha A_\gamma \\
&= \epsilon^{\alpha\beta\gamma} \, (\partial_\alpha \partial_\beta A_\gamma - \partial_\beta \partial_\alpha A_\gamma) \\
&= 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
0 &= \epsilon^{\alpha\beta\gamma} \, F_{\beta\gamma} \\
&= \epsilon^{\alpha\beta\gamma} \, \partial_\beta A_\gamma - \epsilon^{\alpha\beta\gamma} \, \partial_\gamma A_\beta \\
&= \epsilon^{\alpha\beta\gamma} \, \partial_\beta A_\gamma + \epsilon^{\alpha\beta\gamma} \, \partial_\beta A_\gamma \\
&= 2 \, \epsilon^{\alpha\beta\gamma} \, \partial_\beta A_\gamma
\end{aligned}
$$

$$
\begin{aligned}
\implies \; & \epsilon^{\alpha\beta\gamma} \, \partial_\beta A_\gamma = 0 \text{ for any } \alpha \\
& \text{and } \partial_\beta A_\gamma = \partial_\gamma A_\beta \text{ for any } \beta, \gamma
\end{aligned}
$$

$$
\mathcal{L}_{\text{CS}} = \epsilon^{\lambda\mu\nu} \, A_\lambda \, \partial_\mu A_\nu
$$

$$
\begin{aligned}
T_{\text{CS}}{}^{\beta\alpha} &= \frac{\partial \mathcal{L}_{\text{CS}}}{\partial(\partial_\beta A_\gamma)} \, \partial^\alpha A_\gamma - \delta^{\beta\alpha} \, \mathcal{L}_{\text{CS}} \\
&= \epsilon^{\lambda\mu\nu} \, A_\lambda \, \delta^\beta{}_\mu \, \delta^\gamma{}_\nu \, \partial^\alpha A_\gamma - \delta^{\beta\alpha} \, \epsilon^{\lambda\mu\nu} \, A_\lambda \, \partial_\mu A_\nu \\
T_{\text{CS}}{}^{\beta\alpha} &= \epsilon^{\lambda\beta\nu} \, A_\lambda \, \partial^\alpha A_\nu && \text{(since } \epsilon^{\lambda\mu\nu} \, \partial_\mu A_\nu = 0\text{)}
\end{aligned}
$$

(e)

$$
\begin{aligned}
T_{\rm CS}{}^{00} &= A_2\,\partial^0 A_1 - A_1\,\partial^0 A_2 \\
&= A_2\,\partial_0 A_1 - A_1\,\partial_0 A_2 && \text{(since } \partial^0 = g^{\alpha 0}\,\partial_\alpha = \partial_0\text{)} \\
\int d^3x\, T_{\rm CS}{}^{00} &= \int d^3x\, A_2\,\partial_0 A_1 - \int d^3x\, A_1\,\partial_0 A_2 \\
&= \int d^3x\, A_2\,\partial_1 A_0 - \int d^3x\, A_1\,\partial_2 A_0 && \text{(since } \partial_\beta A_\gamma = \partial_\gamma A_\beta\text{)} \\
&= \left. A_2\, A_0\right|_{-\infty}^{\infty} - \int d^3x\, A_0\,\partial_1 A_2 - \left. A_1\, A_0\right|_{-\infty}^{\infty} + \int d^3x\, A_0\,\partial_2 A_1 && \text{(integrating by parts)} \\
&= 0 - 0 + \int d^3x\, A_0\,(\partial_2 A_1 - \partial_1 A_2) && \text{(neglecting surface terms at infinity)} \\
&= 0 && \text{(since } \partial_\beta A_\gamma = \partial_\gamma A_\beta\text{)}
\end{aligned}
$$

3.

$$\begin{aligned}\mathcal{L}_E &= -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu}\\ &= -\frac{1}{4}\,g^{\lambda\mu}\,g^{\rho\nu}\,F_{\mu\nu}\,F_{\lambda\rho}\\ &= -\frac{1}{4}\,g^{\lambda\mu}\,g^{\rho\nu}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\partial_\lambda A_\rho - \partial_\rho A_\lambda)\end{aligned}$$

$$\begin{aligned}T^{\alpha\beta} &= \frac{\partial\mathcal{L}_E}{\partial(\partial_\alpha A_\gamma)}\,\partial^\beta A_\gamma - \delta^{\alpha\beta}\,\mathcal{L}_E\\ &= -\frac{1}{4}\,g^{\lambda\mu}\,g^{\rho\nu}\,((\delta^\alpha{}_\mu\,\delta^\gamma{}_\nu - \delta^\alpha{}_\nu\,\delta^\gamma{}_\mu)\,(\partial_\lambda A_\rho - \partial_\rho A_\lambda) + (\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\delta^\alpha{}_\lambda\,\delta^\gamma{}_\rho - \delta^\alpha{}_\rho\,\delta^\gamma{}_\lambda))\,\partial^\beta A_\gamma + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}\\ &= -\frac{1}{4}\left((\delta^\alpha{}_\mu\,\delta^\gamma{}_\nu - \delta^\alpha{}_\nu\,\delta^\gamma{}_\mu)\,F^{\mu\nu} + (\delta^\alpha{}_\lambda\,\delta^\gamma{}_\rho - \delta^\alpha{}_\rho\,\delta^\gamma{}_\lambda)\,F^{\lambda\rho}\right)\partial^\beta A_\gamma + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}\\ &= -\frac{1}{4}\,(F^{\alpha\gamma} - F^{\gamma\alpha} + F^{\alpha\gamma} - F^{\gamma\alpha})\,\partial^\beta A_\gamma + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}\\ &= F^{\gamma\alpha}\,\partial^\beta A_\gamma + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\text{Consider }\Theta^{\alpha\beta} &= T^{\alpha\beta} - \partial_\gamma\big(F^{\gamma\alpha}\,A^\beta\big)\\ \partial_\gamma\big(F^{\gamma\alpha}\,A^\beta\big) &= \partial_\gamma\big(-F^{\alpha\gamma}\,A^\beta\big)\\ &= -\partial_\gamma\big(F^{\alpha\gamma}\,A^\beta\big)\end{aligned}$$

Thus the term added to $T^{\alpha\beta}$ to get $\Theta^{\alpha\beta}$ is anti-symmetric in α and γ, and thus $\Theta^{\alpha\beta}$ can also be considered a stress-energy tensor.

$$\begin{aligned}\Theta^{\alpha\beta} &= T^{\alpha\beta} - \partial_\gamma\big(F^{\gamma\alpha}\,A^\beta\big)\\ &= F^{\gamma\alpha}\,\partial^\beta A_\gamma + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu} - \partial_\gamma F^{\gamma\alpha}\,A^\beta - F^{\gamma\alpha}\,\partial_\gamma A^\beta\\ &= F^{\gamma\alpha}\left(\partial^\beta A_\gamma - \partial_\gamma A^\beta\right) + \frac{1}{4}\,\delta^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu} - 0 && (\text{since } \partial_\gamma F^{\gamma\alpha} = 0 \text{ from e.o.m.})\\ \Theta^{\alpha\beta} &= F^{\gamma\alpha}\,F^\beta{}_\gamma + \frac{1}{4}\,g^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu} && (\text{since } \delta^{\alpha\beta} = g^{\alpha\gamma}\,\delta_\gamma{}^\beta = g^{\alpha\beta})\end{aligned}$$

$$\begin{aligned}F_{\mu\nu}\,F^{\mu\nu} &= F_{00}\,F^{00} + F_{i0}\,F^{i0} + F_{0j}\,F^{0j} + F_{ij}\,F^{ij}\\ &= 0 + (-E_i)\left(E^i\right) + (E_j)\left(-E^j\right) + \left(-\epsilon_{ijk}\,B^k\right)\left(-\epsilon^{ijl}\,B_l\right)\\ &= -2\,E_i\,E^i + \epsilon_{ijk}\,\epsilon^{ijl}\,B^k\,B_l\\ &= -2\,E_i\,E^i + 2\,\delta^l{}_k\,B^k\,B_l\\ &= 2\left(\vec{B}^2 - \vec{E}^2\right)\\ F^\beta{}_\gamma &= -g^{\beta\lambda}\,F^{\alpha\gamma}\end{aligned}$$

$$\implies \Theta^{\alpha\beta} = -g^{\beta\lambda}\,F^{\alpha\gamma}\,F_{\lambda\gamma} + \frac{1}{2}\,g^{\alpha\beta}\left(\vec{B}^2 - \vec{E}^2\right)$$

$$
\begin{aligned}
\Theta^{00} &= -g^{0\lambda}\, F^{0\gamma}\, F_{\lambda\gamma} + \frac{1}{2}\, g^{00} \left(\vec{B}^2 - \vec{E}^2\right) \\
&= -F^{0\gamma}\, F_{0\gamma} + \frac{1}{2} \left(\vec{B}^2 - \vec{E}^2\right) \\
&= -F^{00}\, F_{00} - (-E^{\gamma})\,(E^{\gamma}) + \frac{1}{2} \left(\vec{B}^2 - \vec{E}^2\right) \\
&= 0 + \vec{E}^2 + \frac{1}{2} \left(\vec{B}^2 - \vec{E}^2\right) \\
&= \frac{1}{2} \left(\vec{B}^2 + \vec{E}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\Theta^{ij} &= -g^{j\lambda}\, F^{i\gamma}\, F_{\lambda\gamma} + \frac{1}{2}\, g^{ij} \left(\vec{B}^2 - \vec{E}^2\right) \\
&= F^{i\gamma}\, F_{j\gamma} - \frac{1}{2}\, \delta^i{}_j \left(\vec{B}^2 - \vec{E}^2\right) \\
&= F^{i0}\, F_{j0} + F^{ik}\, F_j k - \frac{1}{2}\, \delta^i_j\, B_k\, B^k + \frac{1}{2}\, \delta^i{}_j\, \vec{E}^2 \\
&= E^i \left(-E^j\right) + \left(-\epsilon^{ikl}\, B_l\right) \left(-\epsilon_{jkm}\, B^m\right) - \frac{1}{2}\, \delta^i_j\, B_k\, B^k + \frac{1}{2}\, \delta^i{}_j\, \vec{E}^2 \\
&= -E^i\, E^j + \left(\delta^i{}_j\, \delta^l{}_m - \delta^i{}_m\, \delta^l{}_j\right) B_l\, B^m - \frac{1}{2}\, \delta^i_j\, B_k\, B^k + \frac{1}{2}\, \delta^i{}_j\, \vec{E}^2 \\
&= -E^i E^j + \delta^i{}_j\, B_l\, B^l - B_j\, B^i - \frac{1}{2}\, \delta^i_j\, B_k\, B^k + \frac{1}{2}\, \delta^i{}_j\, \vec{E}^2 \\
&= -E^i E^j - B^i\, B^j + \frac{1}{2}\, \delta^i{}_j \left(\vec{B}^2 + \vec{E}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\Theta^{0i} &= -g^{i\lambda}\, F^{0\gamma}\, F_{\lambda\gamma} + \frac{1}{2}\, \delta^0{}_i \left(\vec{B}^2 - \vec{E}^2\right) \\
&= F^{0\gamma}\, F_{i\gamma} + 0 \\
&= F^{00}\, F_{i0} + F^{0j}\, F_{ij} \\
&= 0 - E^j \left(-\epsilon_{ijk}\, B^k\right) \\
&= \epsilon_{ijk}\, E^j\, B^k \\
&= \left(\vec{E} \times \vec{B}\right)_i
\end{aligned}
$$

$$
\begin{aligned}
\Theta^{i0} &= -g^{0\lambda}\, F^{i\gamma}\, F_{\lambda\gamma} + \frac{1}{2}\, \delta^i{}_0 \left(\vec{B}^2 - \vec{E}^2\right) \\
&= -F^{i\gamma}\, F_{0\gamma} + 0 \\
&= -F^{i0}\, F_{00} - F^{ij}\, F_{0j} \\
&= 0 + \epsilon^{ijk}\, B_k\, E_j \\
&= \epsilon^{ijk}\, E_j B_k \\
&= \left(\vec{E} \times \vec{B}\right)_i
\end{aligned}
$$

$$
\Theta^{00} = \frac{1}{2} \left(\vec{B}^2 + \vec{E}^2\right) \qquad \Theta^{0i} = \Theta^{i0} = \left(\vec{E} \times \vec{B}\right)_i \qquad \Theta^{ij} = -E^i E^j - B^i\, B^j + \frac{1}{2}\, \delta^i{}_j \left(\vec{B}^2 + \vec{E}^2\right)
$$