MAU34401: Classical Field Theory
Homework 3 due 05/11/2021

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JS Theoretical Physics

1.

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{4\pi\,\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \left(\Phi(\vec{x}')\,\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}')\,\frac{\partial \Phi(\vec{x}')}{\partial n'}\right) da' \\
&= \frac{A}{4\pi}\oint_S \cos\theta'\,\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,da',
\end{aligned}$$

Since $\Phi(\vec{x}) = A\cos\theta$ and $G(\vec{x},\vec{x}') = 0$ on the surface, and $\rho(\vec{x}) = 0$. Since the surface is a sphere, the derivative of G can be written as

$$\begin{aligned}
\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} &= \left.\frac{\partial G(\vec{x},\vec{x}')}{\partial r'}\right|_{r'=a} \\
&= \frac{\partial}{\partial r'}\left.\left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{(r')^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1}\,(r')^{l+1}}\right)Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\right)\right|_{r'=a} \qquad \begin{matrix} r = r_> \\ r' = r_< \end{matrix} \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{l\,a^{l-1}}{r^{l+1}} + \frac{(l+1)\,a^{2l+1}}{r^{l+1}\,a^{l+2}}\right)Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{(2l+1)\,r^{l+1}}\left(l\,a^{l-1} + l\,a^{l-1} + a^{l-1}\right)Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi\,a^{l-1}}{r^{l+1}}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)
\end{aligned}$$

$$\begin{aligned}
\implies \Phi(\vec{x}) &= A\oint_S \cos\theta'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\,da' \\
&= A\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{l-1}}{r^{l+1}}\,a^2\cos\theta'\sin\theta' Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi) \\
&= A\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1}Y_{lm}(\theta,\phi)\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\cos\theta'\sin\theta'\,Y^*_{lm}(\theta',\phi')
\end{aligned}$$

$$\begin{aligned}
Y_{lm}(\theta,\phi) &\equiv \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_{lm}(\cos\theta)\,e^{i\,m\,\phi} \\
Y_{10}(\theta',\phi') &= \sqrt{\frac{2+1}{4\pi}\,\frac{1!}{1!}}\,P_{10}(\cos\theta')\,e^0 \\
&= \sqrt{\frac{3}{4\pi}}\,\cos\theta' \\
\implies \cos\theta' &= \sqrt{\frac{4\pi}{3}}\,Y_{10}(\theta',\phi')
\end{aligned}$$

$$
\begin{aligned}
\implies \Phi(\vec{x}) &= A \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\phi)\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\phi')\sin\theta'\, Y_{lm}^*(\theta',\phi') \\
&= A\sqrt{\frac{4\pi}{3}} \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\phi)\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'\, Y_{lm}^*(\theta',\phi')\, Y_{10}(\theta',\phi') \\
&= A\sqrt{\frac{4\pi}{3}} \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\phi)\,\delta_{l1}\,\delta_{m0} \\
&= A\sqrt{\frac{4\pi}{3}}\left(\frac{a}{r}\right)^{2} Y_{10}(\theta,\phi) \\
&= A\sqrt{\frac{4\pi}{3}}\left(\frac{a}{r}\right)^{2}\sqrt{\frac{3}{4\pi}}\,\cos\theta \\
\Phi(\vec{x}) &= A\left(\frac{a}{r}\right)^{2}\cos\theta
\end{aligned}
$$

$$
\Phi(\vec{x})\big|_{r=a} = A\cos\theta \qquad\qquad \lim_{r\to\infty}\Phi(\vec{x}) = 0,
$$

and thus the boundary conditions are met.

2.

Say we are given a Green's function $G(\vec{x},\vec{x}')$ satisfying Neumann boundary conditions. If we consider a new Green's function $\mathcal{G}(\vec{x},\vec{x}') = G(\vec{x},\vec{x}') + F(\vec{x})$, we can show that this satisfies the same boundary conditions since $\nabla^{2\prime}\mathcal{G}(\vec{x},\vec{x}') = \nabla^{2\prime}G(\vec{x},\vec{x}') + \nabla^{2\prime}F(\vec{x}) = \nabla^{2\prime}G(\vec{x},\vec{x}')$, and $\frac{\partial\mathcal{G}(\vec{x},\vec{x}')}{\partial n'} = \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} + \frac{\partial F(\vec{x})}{\partial n'} = \frac{\partial G(\vec{x},\vec{x}')}{\partial n'}$. From Green's second identity, we have

$$
\begin{aligned}
\int_V \left(\mathcal{G}(\vec{x},\vec{x}')\,\nabla^2\mathcal{G}(\vec{x}',\vec{x}) - \mathcal{G}(\vec{x}',\vec{x})\,\nabla^2\mathcal{G}(\vec{x},\vec{x}')\right) d^3x &= \oint_S \left(\mathcal{G}(\vec{x},\vec{x}')\,\frac{\partial\mathcal{G}(\vec{x}',\vec{x})}{\partial n} - \mathcal{G}(\vec{x}',\vec{x})\,\frac{\partial\mathcal{G}(\vec{x},\vec{x}')}{\partial n}\right) da \\
-4\pi\int_V \left(\mathcal{G}(\vec{x},\vec{x}') - \mathcal{G}(\vec{x}',\vec{x})\right)\delta^{(3)}(\vec{x}-\vec{x}')\, d^3x &= \oint_S \left(\mathcal{G}(\vec{x},\vec{x}')\,\frac{\partial\mathcal{G}(\vec{x}',\vec{x})}{\partial n} - \mathcal{G}(\vec{x}',\vec{x})\,\frac{\partial\mathcal{G}(\vec{x},\vec{x}')}{\partial n}\right) da \\
\mathcal{G}(\vec{x},\vec{x}') - \mathcal{G}(\vec{x}',\vec{x}) &= -\frac{1}{4\pi}\oint_S \left(\mathcal{G}(\vec{x},\vec{x}')\,\frac{\partial\mathcal{G}(\vec{x}',\vec{x})}{\partial n} - \mathcal{G}(\vec{x}',\vec{x})\,\frac{\partial\mathcal{G}(\vec{x},\vec{x}')}{\partial n}\right) da
\end{aligned}
$$

If we assume that the Green's function takes the simplest form allowable, i.e. $\frac{\mathcal{G}(\vec{x},\vec{x}')}{\partial n} = -\frac{4\pi}{a}$, where a is the area of the surface in question, we get

$$
\begin{aligned}
\oint_S \left(\mathcal{G}(\vec{x},\vec{x}')\,\frac{\partial\mathcal{G}(\vec{x}',\vec{x})}{\partial n} - \mathcal{G}(\vec{x}',\vec{x})\,\frac{\partial\mathcal{G}(\vec{x},\vec{x}')}{\partial n}\right) da &= -\frac{4\pi}{a}\oint_S \left(\mathcal{G}(\vec{x},\vec{x}') - \mathcal{G}(\vec{x}',\vec{x})\right) da \\
&= -\frac{4\pi}{a}\oint_S \left(G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})\right) da - \frac{4\pi}{a}\oint_S \left(F(\vec{x}) - F(\vec{x}')\right) da \\
&= -\frac{4\pi}{a}\oint_S \left(G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})\right) da - 4\pi\left(F(\vec{x}) - F(\vec{x}')\right)
\end{aligned}
$$

For $\mathcal{G}$ to be symmetric we must have $\mathcal{G}(\vec{x},\vec{x}') - \mathcal{G}(\vec{x}',\vec{x}) = 0$, and so the above expression must be 0. We thus get

$$
\begin{aligned}
F(\vec{x}) - F(\vec{x}') &= -\frac{1}{a}\oint_S \left(G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})\right) da \\
\implies F(\vec{x}) &= -\frac{1}{a}\oint_S G(\vec{x},\vec{x}')\, da
\end{aligned}
$$

Thus, given any Green's function $G(\vec{x},\vec{x}')$ that satisfies Neumann boundary conditions, we can add the term $-\frac{1}{a}\oint_S G(\vec{x},\vec{x}')\, da$ to get another Green's function that satisfies the same conditions that is symmetric under an exchange of its two position variables.

3.

(a)

Since Φ is continuous at the boundary, i.e. when $r = R$, we can equate the two piecewise functions evaluated at the boundary to find a relationship between A_l and B_l.

$$\sum_{l=0}^{\infty} A_l\, R^l\, P_l \cos\theta = \sum_{l=0}^{\infty} B_l\, \frac{1}{R^{l+1}}\, P_l \cos\theta$$
$$\implies B_l = A_l\, R^{2l+1}$$

$$\begin{aligned}
\frac{\sigma(\theta)}{\varepsilon_0}\,\hat{n} &= \vec{E}_{\text{out}} - \vec{E}_{\text{in}}\Big|_{\text{surface}} \\
\frac{\sigma(\theta)}{\varepsilon_0}\,\hat{n}\cdot\hat{n} &= (-\nabla\Phi_{\text{out}} + \nabla\Phi_{\text{in}})\cdot\hat{n}|_{\text{surface}} \\
\frac{\sigma(\theta)}{\varepsilon_0} &= \frac{\partial\Phi_{\text{in}}}{\partial n} - \frac{\partial\Phi_{\text{out}}}{\partial n}\Bigg|_{\text{surface}} \\
&= \frac{\partial\Phi_{\text{in}}}{\partial r} - \frac{\partial\Phi_{\text{out}}}{\partial r}\Bigg|_{r=R} \\
&= \sum_{l=0}^{\infty} l\, A_l\, r^{l-1}\, P_l(\cos\theta) - \sum_{l=0}^{\infty}(-l-1)\,\frac{B_l}{r^{l+2}}\, P_l(\cos\theta)\Bigg|_{r=R} \\
&= \sum_{l=0}^{\infty}\left(l\, A_l\, R^{l-1}\, P_l(\cos\theta) + (l+1)\, A_l\, R^{l-1}\, P_l(\cos\theta)\right) \\
&= \sum_{l=0}^{\infty}(2l+1)\, A_l\, R^{l-1}\, P_l(\cos\theta) \\
\frac{1}{\varepsilon_0}\int_0^{\pi}\sigma(\theta)\, P_\lambda(\cos\theta)\,\sin\theta\, d\theta &= \sum_{l=0}^{\infty}(2l+1)\, A_l\, R^{l-1}\int_0^{\pi} P_l(\cos\theta)\, P_\lambda(\cos\theta)\,\sin\theta\, d\theta \\
&\qquad \text{(multiplying across by } P_\lambda(\cos\theta)\sin\theta \text{ and integrating with respect to } \theta \text{ over } [0,\pi]) \\
&= -\sum_{l=0}^{\infty}(2l+1)\, A_l\, R^{l-1}\int_1^{-1} P_l(x)\, P_\lambda(x)\, dx \qquad \begin{pmatrix} x = \cos\theta \\ dx = -\sin\theta\, d\theta \end{pmatrix} \\
&= \sum_{l=0}^{\infty}(2l+1)\, A_l\, R^{l-1}\int_{-1}^{1} P_l(x)\, P_\lambda(x)\, dx \\
&= \sum_{l=0}^{\infty}(2l+1)\, A_l\, R^{l-1}\,\frac{2\delta_{l\lambda}}{2l+1} \qquad \text{(using } \textstyle\int_{-1}^{1} P_l(x)\, P_m(x)\, dx = \frac{2\delta_{lm}}{2l+1}) \\
&= 2A_\lambda\, R^{\lambda-1} \\
\implies A_\lambda &= \frac{1}{2R^{\lambda-1}\,\varepsilon_0}\int_0^{\pi}\sigma(\theta)\, P_\lambda(\cos\theta)\,\sin\theta\, d\theta
\end{aligned}$$

In Gaussian units, where $\frac{1}{4\pi\varepsilon_0} = 1$, the above expression is equivalent to

$$A_l = \frac{4\pi}{2R^{l-1}}\int_0^{\pi}\sigma(\theta)\, P_l(\cos\theta)\,\sin\theta\, d\theta$$

(b)

$$\begin{aligned}
A_l &= \frac{4k\,\pi}{2R^{l-1}} \int_0^{\pi} d\theta \sin\theta\cos\theta\, P_l(\cos\theta) \\
&= -\frac{4k\,\pi}{2R^{l-1}} \int_1^{-1} x\, P_l(x)\, dx && \begin{pmatrix} x = \cos\theta \\ dx = -\sin\theta\, d\theta \end{pmatrix} \\
&= \frac{4k\,\pi}{2R^{l-1}} \int_{-1}^{1} P_1(x)\, P_l(x)\, dx \\
&= \frac{4k\,\pi}{2R^{l-1}} \frac{2\delta_{1l}}{2(1)+1} && \left(\text{using } \int_{-1}^{1} P_l(x)\, P_m(x)\, dx = \tfrac{2\delta_{lm}}{2l+1}\right) \\
&= \frac{4k\,\pi}{3R^{l-1}}\, \delta_{1l}
\end{aligned}$$

$$A_l = \begin{cases} \dfrac{4k\,\pi}{3}, & l = 1 \\[2ex] 0, & l \neq 1 \end{cases} \implies B_l = \begin{cases} \dfrac{4k\,\pi\,R^3}{3}, & l = 1 \\[2ex] 0, & l \neq 1 \end{cases}$$

$$\implies \Phi(r,\theta) = \begin{cases} \dfrac{4k\,\pi}{3}\, r\cos\theta, & r < R \\[2ex] \dfrac{4k\,\pi\,R^3}{3} \dfrac{\cos\theta}{r^2}, & r > R \end{cases}$$

$$\begin{aligned}
\vec{E}(r,\theta) &\equiv -\nabla\Phi(r,\theta) \\
&= -\frac{\partial\Phi}{\partial r}\,\hat{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\hat{\theta} - \frac{1}{r\,\sin\theta}\frac{\partial\Phi}{\partial\varphi}\,\hat{\varphi} \\
&= \begin{cases} -\dfrac{4k\,\pi}{3}\cos\theta\,\hat{r} + \dfrac{4k\,\pi}{3}\sin\theta\,\hat{\theta} + 0, & r < R \\[2ex] \dfrac{8k\,\pi\,R^3}{3}\dfrac{\cos\theta}{r^3}\,\hat{r} + \dfrac{4k\,\pi\,R^3}{3}\dfrac{\sin\theta}{r^3}\,\hat{\theta} + 0, & r > R \end{cases} \\
&= \begin{cases} -\dfrac{4k\,\pi}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right), & r < R \\[2ex] \dfrac{4k\,\pi\,R^3}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right), & r > R \end{cases} \\
&= \begin{cases} -\dfrac{4k\,\pi}{3}\,\hat{z}, & r < R \\[2ex] \dfrac{4k\,\pi\,R^3}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right), & r > R \end{cases}
\end{aligned}$$