MAU34401: Classical Field Theory
Homework 2 due 19/09/2021

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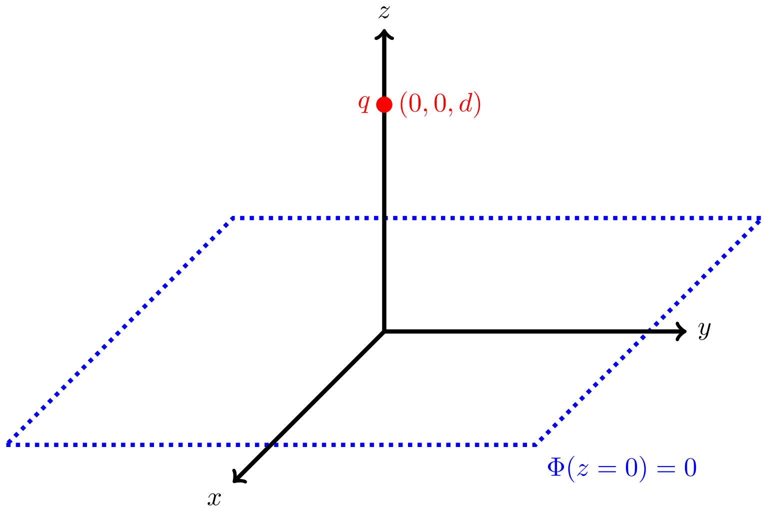
JS Theoretical Physics

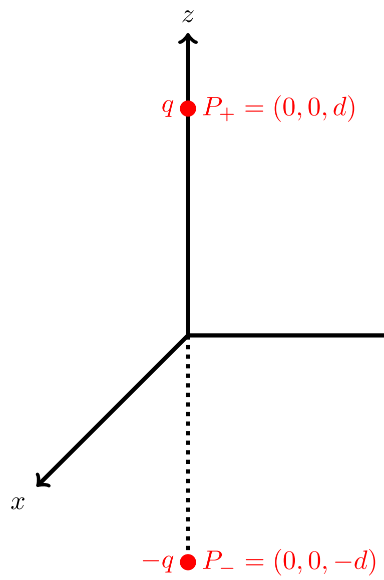
1.

(a)

Original problem



Method of images problem



(b)

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left|\vec{r}-\vec{P}_+\right|} + \frac{-q}{\left|\vec{r}-\vec{P}_-\right|}\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a_+} - \frac{1}{a_-}\right)$$

$$a_+ = \sqrt{(x-0)^2 + (y-0)^2 + (z-d)^2} = \sqrt{x^2+y^2+z^2+d^2-2zd}$$

$$a_- = \sqrt{(x-0)^2 + (y-0)^2 + (z+d)^2} = \sqrt{x^2+y^2+z^2+d^2+2zd}$$

$$\implies \Phi(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+z^2+d^2-2zd}} - \frac{1}{\sqrt{x^2+y^2+z^2+d^2+2zd}}\right)$$

$$\Phi(x,y,0) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+d^2}} - \frac{1}{\sqrt{x^2+y^2+d^2}}\right)$$
$$= 0 \implies \text{boundary condition satisfied}$$

$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q\,(\vec{r}-\vec{p}_+)}{\left|\vec{r}-\vec{p}_+\right|^3} + \frac{-q\,(\vec{r}-\vec{p}_-)}{\left|\vec{r}-\vec{p}_-\right|^3}\right)$$

$$\vec{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left(\frac{x\,\hat{\imath} + y\,\hat{\jmath} + (z-d)\,\hat{k}}{(x^2+y^2+z^2+d^2-2zd)^{\frac{3}{2}}} - \frac{x\,\hat{\imath} + y\,\hat{\jmath} + (z+d)\,\hat{k}}{(x^2+y^2+z^2+d^2+2zd)^{\frac{3}{2}}}\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(x\left(\frac{1}{(x^2+y^2+z^2+d^2-2zd)^{\frac{3}{2}}} - \frac{1}{(x^2+y^2+z^2+d^2+2zd)^{\frac{3}{2}}}\right)\hat{\imath}\right.$$
$$+y\left(\frac{1}{(x^2+y^2+z^2+d^2-2zd)^{\frac{3}{2}}} - \frac{1}{(x^2+y^2+z^2+d^2+2zd)^{\frac{3}{2}}}\right)\hat{\jmath}$$
$$+\left[z\left(\frac{1}{(x^2+y^2+z^2+d^2-2zd)^{\frac{3}{2}}} - \frac{1}{(x^2+y^2+z^2+d^2+2zd)^{\frac{3}{2}}}\right)\right.$$
$$\left.\left. -\ d\left(\frac{1}{(x^2+y^2+z^2+d^2-2zd)^{\frac{3}{2}}} + \frac{1}{(x^2+y^2+z^2+d^2+2zd)^{\frac{3}{2}}}\right)\right]\hat{k}\right)$$

$$\sigma(x,y,z=0) = -\varepsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{z=0}$$

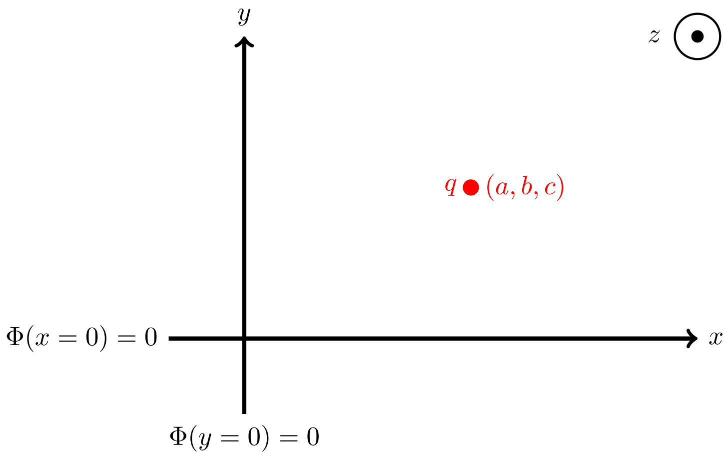
$$\vec{E} = -\frac{\partial\Phi}{\partial r}\,\hat{r} \implies \frac{\partial\Phi}{\partial r} = \left|\vec{E}\right|$$

$$\implies \sigma(x,y,z=0) = -\varepsilon_0\left|\vec{E}(x,y,z=0)\right|$$
$$= -\frac{q\varepsilon_0}{4\pi\varepsilon_0}\left(\sqrt{0+0+d^2\left(\frac{1}{(x^2+y^2+d^2)^{\frac{3}{2}}} + \frac{1}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right)^2}\right)$$

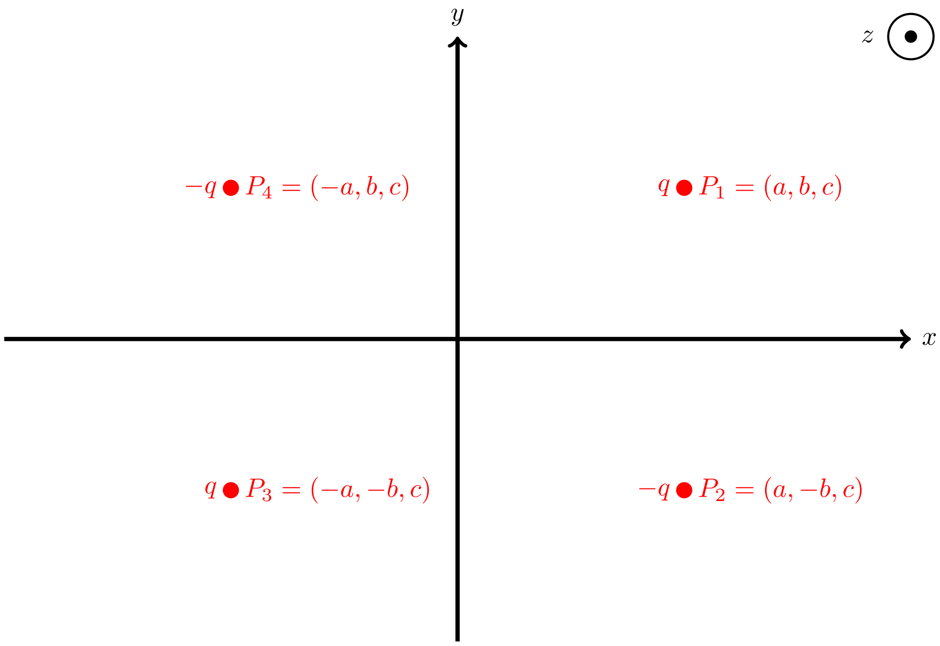
$$\sigma(x,y,z=0) = -\frac{qd}{2\pi}\frac{1}{(x^2+y^2+d^2)^{\frac{3}{2}}}$$

(c)

Original problem (x-z and y-z planes are considered as the grounded perpendicular conducting planes, and the point q is considered at an arbitrary z point c)



Method of images problem



$$\begin{aligned}\Phi(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\left|\vec{r}-\vec{P}_1\right|} + \frac{-q}{\left|\vec{r}-\vec{P}_2\right|} + \frac{q}{\left|\vec{r}-\vec{P}_3\right|} + \frac{-q}{\left|\vec{r}-\vec{P}_4\right|}\right)\\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right)\end{aligned}$$

$$\begin{aligned}r_1 &= \sqrt{(x-a)^2 + (y-b)^2 + (z-d)^2}\\ &= \sqrt{x^2+y^2+z^2+a^2+b^2+c^2-2xa-2yb-2zd}\\ r_3 &= \sqrt{(x+a)^2 + (y+b)^2 + (z-d)^2}\\ &= \sqrt{x^2+y^2+z^2+a^2+b^2+c^2+2xa+2yb-2zd}\end{aligned}$$

$$\begin{aligned}r_2 &= \sqrt{(x-a)^2 + (y+b)^2 + (z-d)^2}\\ &= \sqrt{x^2+y^2+z^2+a^2+b^2+c^2-2xa+2yb-2zd}\\ r_4 &= \sqrt{(x+a)^2 + (y-b)^2 + (z-d)^2}\\ &= \sqrt{x^2+y^2+z^2+a^2+b^2+c^2+2xa-2yb-2zd}\end{aligned}$$

For convenience label $\lambda = x^2 + y^2 + z^2$, $\chi = a^2 + b^2 + c^2$, $\alpha = 2xa$, $\beta = 2yb$, $\gamma = 2zd$

$$r_1 = \sqrt{\lambda + \chi - \alpha - \beta - \gamma} \qquad r_2 = \sqrt{\lambda + \chi - \alpha + \beta - \gamma}$$
$$r_3 = \sqrt{\lambda + \chi + \alpha + \beta - \gamma} \qquad r_4 = \sqrt{\lambda + \chi + \alpha - \beta - \gamma}$$

$$\begin{aligned}
\implies \Phi(x,y,z) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r+p-\alpha-\beta-\gamma}} - \frac{1}{\sqrt{r+p-\alpha+\beta-\gamma}}\right. \\
&\quad \left. + \frac{1}{\sqrt{r+p+\alpha+\beta-\gamma}} - \frac{1}{\sqrt{r+p+\alpha-\beta-\gamma}}\right) \\
\Phi(0,y,z) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r+p-\beta-\gamma}} - \frac{1}{\sqrt{r+p+\beta-\gamma}}\right. \\
&\quad \left. + \frac{1}{\sqrt{r+p+\beta-\gamma}} - \frac{1}{\sqrt{r+p-\beta-\gamma}}\right) \\
&= 0 \\
\Phi(x,0,z) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r+p-\alpha-\gamma}} - \frac{1}{\sqrt{r+p-\alpha-\gamma}}\right. \\
&\quad \left. + \frac{1}{\sqrt{r+p+\alpha-\gamma}} - \frac{1}{\sqrt{r+p+\alpha-\gamma}}\right) \\
&= 0 \implies \text{boundary conditions satisfied}
\end{aligned}$$

Consider the electric field generated by the image charges only

$$\begin{aligned}
\vec{E}'(x,y,z) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{-q\left(\left(x-a\right)\hat{\imath}\right) + \left(y+b\right)\hat{\jmath} + \left(z-c\right)\hat{k}\right)}{\left(r+p-\alpha+\beta-\gamma\right)^{\frac{3}{2}}} + \frac{q\left(\left(x+a\right)\hat{\imath}\right) + \left(y+b\right)\hat{\jmath} + \left(z-c\right)\hat{k}\right)}{\left(r+p+\alpha+\beta-\gamma\right)^{\frac{3}{2}}}\right. \\
&\quad \left. + \frac{-q\left(\left(x+a\right)\hat{\imath}\right) + \left(y-b\right)\hat{\jmath} + \left(z-c\right)\hat{k}\right)}{\left(r+p+\alpha-\beta-\gamma\right)^{\frac{3}{2}}}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(-\frac{(x-a)\,\hat{\imath} + (y+b)\,\hat{\jmath} + (z-c)\,\hat{k}}{\left(r+p-\alpha+\beta-\gamma\right)^{\frac{3}{2}}} + \frac{(x+a)\,\hat{\imath} + (y+b)\,\hat{\jmath} + (z-c)\,\hat{k}}{\left(r+p+\alpha+\beta-\gamma\right)^{\frac{3}{2}}}\right. \\
&\quad \left. - \frac{(x+a)\,\hat{\imath} + (y-b)\,\hat{\jmath} + (z-c)\,\hat{k}}{\left(r+p+\alpha-\beta-\gamma\right)^{\frac{3}{2}}}\right)
\end{aligned}$$

Now consider $\vec{E}'(a,b,c)$

$$(r+p)\,|_{(a,b,c)} = 2\left(a^2+b^2+c^2\right)$$
$$\alpha|_{(a,b,c)} = 2a^2 \qquad \beta|_{(a,b,c)} = 2b^2 \qquad \gamma|_{(a,b,c)} = 2c^2$$

$$\begin{aligned}
\implies \vec{E}'(a,b,c) &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{2b\,\hat{\jmath}}{\left(4b^2\right)^{\frac{3}{2}}} + \frac{2a\,\hat{\imath} + 2b\,\hat{\jmath}}{\left(4a^2+4b^2\right)^{\frac{3}{2}}} - \frac{2a\,\hat{\imath}}{\left(4a^2\right)^{\frac{3}{2}}}\right) \\
&= \frac{q}{16\pi\varepsilon_0}\left(\left(\frac{a}{\left(a^2+b^2\right)^{\frac{3}{2}}} - \frac{1}{a^2}\right)\hat{\imath} + \left(\frac{b}{\left(a^2+b^2\right)^{\frac{3}{2}}} - \frac{1}{b^2}\right)\hat{\jmath}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{F}(a,b,c) &= q\vec{E}'(a,b,c) \\
&= \frac{q^2}{16\pi\varepsilon_0}\left(\left(\frac{a}{\left(a^2+b^2\right)^{\frac{3}{2}}} - \frac{1}{a^2}\right)\hat{\imath} + \left(\frac{b}{\left(a^2+b^2\right)^{\frac{3}{2}}} - \frac{1}{b^2}\right)\hat{\jmath}\right)
\end{aligned}$$

For convenience, we will label the normal vector to each conducting plate as $\hat{e}_a$ and $\hat{e}_b$, respectively. The force acting on q has a magnitude of $\frac{q^2}{16\pi\varepsilon_0}\left(\frac{a}{(a^2+b^2)^{\frac{3}{2}}} - \frac{1}{a^2}\right)$ in the direction of $\hat{e}_a$, and a magnitude of $\frac{q^2}{16\pi\varepsilon_0}\left(\frac{b}{(a^2+b^2)^{\frac{3}{2}}} - \frac{1}{b^2}\right)$ in the direction of $\hat{e}_b$.

2.

$$
\begin{aligned}
\int_V \left(\phi\,\nabla^2\psi - \psi\,\nabla^2\phi\right) d^3x &= \oint_S \left(\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right) da \qquad \text{(Green's second identity)}\\
\int_V \left(G(\vec{x},\vec{x}')\,\nabla^2 G(\vec{x}',\vec{x}) - G(\vec{x}',\vec{x})\,\nabla^2 G(\vec{x},\vec{x}')\right) d^3x &= \oint_S \left(G(\vec{x},\vec{x}')\,\frac{\partial G(\vec{x}',\vec{x})}{\partial n} - G(\vec{x}',\vec{x})\,\frac{\partial G(\vec{x},\vec{x}')}{\partial n}\right) da\\
&\qquad \text{(Labelling } \phi = G(\vec{x},\vec{x}') \text{ and } \psi = G(\vec{x}',\vec{x})\text{)}\\
-4\pi \int_V \left(G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})\right)\delta^{(3)}(\vec{x}-\vec{x}')\,d^3x &= \oint_S (0-0)\,da\\
&\left(\nabla^2 G(\vec{x},\vec{x}') = \nabla^2 G(\vec{x}',\vec{x}) = \delta^{(3)}(\vec{x}-\vec{x}'),\ G(\vec{x},\vec{x}')|_{\vec{x}\in S} = G(\vec{x}',\vec{x})|_{\vec{x}\in S} = 0\right)\\
-4\pi\left(G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})\right) &= 0\\
\implies G(\vec{x},\vec{x}') &= G(\vec{x}',\vec{x})
\end{aligned}
$$

3.

$$
\begin{aligned}
\int_V \nabla\cdot\vec{A}\,d^3x &= \oint_S \vec{A}\cdot\vec{n}\,da \qquad \text{(Gauss's theorem)}\\
\int_V \nabla^2 G(\vec{x},\vec{x}')\,d^3x &= \oint_S \nabla G(\vec{x},\vec{x}')\cdot\vec{n}\,da \qquad \text{(considering } \vec{A} = \nabla G(\vec{x},\vec{x}')\text{)}\\
\int_V -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')\,d^3x &= \oint_S \frac{\partial G(\vec{x},\vec{x}')}{\partial n}\,da\\
-4\pi &= \oint_S \frac{\partial G(\vec{x},\vec{x}')}{\partial n}\,da
\end{aligned}
$$

If we have that $\frac{\partial G(\vec{x},\vec{x}')}{\partial n}$ is a constant, then the above expression is still valid. Substituting $\frac{\partial G(\vec{x},\vec{x}')}{\partial n} = A$ gives us

$$
\begin{aligned}
&\oint_S A\,da = -4\pi\\
&\implies A = -\frac{4\pi}{a} \quad \text{where } a \text{ is the area of the surface in question}
\end{aligned}
$$