MAU34401: Classical Field Theory
Homework 1 due 30/09/2021

Ruaidhrí Campion
19333850
JS Theoretical Physics

1.

The electric field $\vec{E}$ generated by a point charge q is

$$\vec{E} = \frac{q}{4\pi\,\epsilon_0\,r^2}\,\hat{r} \implies \Phi_{\text{flux}} \equiv \oint_S \vec{E}\cdot d\vec{S}$$
$$= \frac{q}{4\pi\,\epsilon_0}\oint_S \frac{1}{r^2}\,\hat{r}\cdot d\vec{S}$$
$$= \frac{q}{4\pi\,\epsilon_0}\oint_S \frac{1}{r^2}\,dA$$
$$= \frac{q}{4\pi\,\epsilon_0}\oint_S d\Omega$$
$$= \frac{q}{\epsilon_0}$$

2.

Writing $r \equiv |\vec{x} - \vec{x}'|$ gives us

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{r}\,d^3x'$$
$$\implies \nabla^2\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,\nabla^2\left(\frac{1}{r}\right)d^3x'.$$

The divergence theorem gives us

$$\int_V \nabla^2\left(\frac{1}{r}\right)d^3x' = \oint_S \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot\vec{n}'\,da'$$
$$= \oint_S \nabla\left(\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{1}{2}}\right)\cdot\vec{n}'\,da'$$
$$= \oint_S -\frac{1}{2}\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{-\frac{3}{2}}\left(2(x-x')\hat{\imath}+2(y-y')\hat{\jmath}+2(z-z')\hat{k}\right)\cdot\vec{n}'\,da'$$
$$= -\oint_S \frac{\vec{r}}{r^3}\cdot\vec{n}'\,da'$$
$$= -\oint_S \frac{r\cos\theta}{r^3}\,da'$$
$$= -\oint_S \frac{1}{r^2}\,r^2\,d\Omega$$
$$= -4\pi.$$

We also have that

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{1}{r^2}\right),$$

which vanishes when $r \neq 0$. Since $\nabla^2\left(\frac{1}{r}\right)$ is 0 when $r \neq 0$ yet we get a finite result when integrating it over any volume containing $r = 0$, we can express it as a delta function as

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(r).$$

We can thus calculate $\nabla^2\Phi$ as

$$
\begin{aligned}
\nabla^2\Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\left(-4\pi\delta^3(r)\right)d^3x' \\
&= -\frac{1}{\epsilon_0}\int_V \rho(\vec{x}')\delta^3(|\vec{x}-\vec{x}'|)\,d^3x' \\
&= -\frac{\rho(\vec{x})}{\epsilon_0},
\end{aligned}
$$

and so $\Phi(\vec{x})$ is a solution of the Poisson equation.

3.

$$
\begin{aligned}
\int_0^1\left[\phi\,\frac{d^2\psi}{dx^2}-\psi\,\frac{d^2\phi}{dx^2}\right]dx &= \int_0^1\left[\phi\,\frac{d^2\psi}{dx^2}+\frac{d\phi}{dx}\frac{d\psi}{dx}-\frac{d\phi}{dx}\frac{d\psi}{dx}-\psi\,\frac{d^2\phi}{dx^2}-\frac{d\psi}{dx}\frac{d\phi}{dx}+\frac{d\psi}{dx}\frac{d\phi}{dx}\right]dx \\
&= \int_0^1\left[\frac{d}{dx}\left(\phi\,\frac{d\psi}{dx}\right)-\frac{d}{dx}\left(\psi\,\frac{d\phi}{dx}\right)+\frac{d\psi}{dx}\frac{d\phi}{dx}-\frac{d\phi}{dx}\frac{d\psi}{dx}\right]dx \\
&= \left[\phi\,\frac{d\psi}{dx}-\psi\,\frac{d\phi}{dx}\right]\Bigg|_0^1
\end{aligned}
$$

Since the volume does not contain any charges, the first integral of the potential $\Phi(\vec{x})$ vanishes as the charge distribution $\rho(x')$ is 0 inside the volume. For a general point $\vec{x}$ this gives us

$$
\Phi(\vec{x}) = \frac{1}{4\pi}\oint_S\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'}-\Phi\,\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'.
$$

If we now want to consider $\Phi(\vec{x}_0)$, we must evaluate the RHS of this expression at $R = R_0$, as the surface is simply the sphere of radius R_0 centred at x_0. We can write the first term in the integral as

$$
\frac{1}{R}\frac{\partial\Phi}{\partial n'}\bigg|_{R=R_0} = \frac{1}{R_0}\nabla\Phi\cdot\vec{n}' = \frac{1}{R_0}\vec{E}\cdot\vec{n}'.
$$

Since we are concerned with the potential at the centre of the sphere over which we are integrating, the normal vector will always be pointing out of the sphere, i.e. $\frac{\partial}{\partial n'} = \frac{\partial}{\partial R}$. Thus the second term becomes

$$
-\,\Phi\,\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\bigg|_{R=R_0} = -\,\Phi\,\frac{\partial}{\partial R}\left(\frac{1}{R}\right)\bigg|_{R=R_0} = \Phi\,\frac{1}{R_0^2}.
$$

We thus result in

$$
\Phi(\vec{x}_0) = \frac{1}{4\pi R_0}\left(\oint_S \vec{E}\cdot\vec{n}'\,da' + \frac{1}{R_0}\oint_S \Phi\,da'\right).
$$

Using the divergence theorem, the first term can be evaluated as

$$
\begin{aligned}
\oint_S \vec{E}(\vec{x}')\cdot\vec{n}'\,da' &= \int_V \nabla\cdot\vec{E}(\vec{x}')\,d^3x' \\
&= \int_V \frac{\rho(\vec{x}')}{\epsilon_0}\,d^3x' \\
&= 0,
\end{aligned}
$$

and so we get the desired result

$$
\Phi(\vec{x}_0) = \frac{1}{4\pi R_0^2}\oint_S \Phi(\vec{x}')\,da' = \langle\Phi\rangle_S.
$$

4.

$$\begin{aligned}\nabla^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) + \left(\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)\left(\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right) + \frac{\partial^2}{\partial z^2}\end{aligned}$$

$$\begin{aligned} x &= \rho\cos\phi & y &= \rho\sin\phi & z &= z \\ \rho &= \sqrt{x^2+y^2} & \phi &= \arctan\left(\frac{y}{x}\right) & z &= z \end{aligned}$$

$$\begin{aligned}\frac{\partial\rho}{\partial x} &= \frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2x) & \frac{\partial\rho}{\partial y} &= \frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2y) \\ &= \frac{x}{\sqrt{x^2+y^2}} & &= \frac{y}{\sqrt{x^2+y^2}} \\ &= \frac{x}{\rho} & &= \frac{y}{\rho} \\ &= \cos\phi & &= \sin\phi \\ \frac{\partial\phi}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2} & \frac{\partial\phi}{\partial y} &= \frac{x}{x^2+y^2} \\ &= -\frac{y}{x^2+y^2} & &= \frac{x}{\rho^2} \\ &= -\frac{y}{\rho^2} & &= \frac{\cos\phi}{\rho} \\ &= -\frac{\sin\phi}{\rho} & & \end{aligned}$$

$$\begin{aligned}\nabla^2 &= \left[\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\right] + \left[\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right)\right] + \frac{\partial^2}{\partial z^2} \\ &= \left[0 + \cos^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\cos\phi\sin\phi}{\rho^2}\frac{\partial}{\partial\phi} - \frac{\cos\phi\sin\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} + \frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\rho} - \frac{\cos\phi\sin\phi}{\rho}\frac{\partial^2}{\partial\phi\partial\rho} + \frac{\cos\phi\sin\phi}{\rho^2}\frac{\partial}{\partial\phi}\right. \\ &\quad \left. + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] + \left[0 + \sin^2\phi\frac{\partial^2}{\partial\rho^2} - \frac{\cos\phi\sin\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\cos\phi\sin\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} + \frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\rho} + \frac{\cos\phi\sin\phi}{\rho}\frac{\partial^2}{\partial\phi\partial\rho}\right. \\ &\quad \left. - \frac{\cos\phi\sin\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\end{aligned}$$