MAU34402: Classical Electrodynamics
Homework 3 due 17/04/2022

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JS Theoretical Physics

Question 1

1.

$$\begin{aligned}
\mathbf{F} &= q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) && \text{(Lorentz force)} \\
\implies F &= |q|vB && \text{(as } \mathbf{E} = \mathbf{0},\ \mathbf{v} \perp \mathbf{B} \text{ for circular motion)} \\
F &= \frac{mv^2}{r} && \text{(centripetal force (with } m \equiv \gamma m_0\text{))} \\
\implies B &= \frac{mv}{|q|r} \\
&= \frac{p}{|q|r}
\end{aligned}$$

$$\begin{aligned}
E^2 &= p^2c^2 + m_0^2c^4 \\
\implies p &= \frac{\sqrt{E^2 - m_0^2c^4}}{c} \\
\implies B &= \frac{\sqrt{E^2 - m_0^2c^4}}{|q|rc}
\end{aligned}$$

$$B_{\text{electron}} = \frac{\sqrt{(27.5 \cdot 10^9 \cdot 1.60217653 \cdot 10^{-19} \text{ J})^2 - (9.1093826 \cdot 10^{-31} \text{ kg})^2 (2.99792458 \cdot 10^8 \text{ m s}^{-1})^4}}{(1.60217653 \cdot 10^{-19} \text{ C}) \left(\frac{6{,}300 \text{ m}}{2\pi}\right) (2.99792458 \cdot 10^8 \text{ m s}^{-1})}$$

$B_{\text{electron}} \approx 0.09149 \text{ T}$

$$B_{\text{proton}} = \frac{\sqrt{(920 \cdot 10^9 \cdot 1.60217653 \cdot 10^{-19} \text{ J})^2 - (1.67262171 \cdot 10^{-27} \text{ kg})^2 (2.99792458 \cdot 10^8 \text{ m s}^{-1})^4}}{(1.60217653 \cdot 10^{-19} \text{ C}) \left(\frac{6{,}300 \text{ m}}{2\pi}\right) (2.99792458 \cdot 10^8 \text{ m s}^{-1})}$$

$B_{\text{proton}} \approx 3.061 \text{ T}$

2.

$$\begin{aligned}
p_\mu &= \left(\frac{E}{c}, -\mathbf{p}\right) \\
\implies p_\mu p^\mu &= \frac{E^2}{c^2} - \mathbf{p}^2 \\
\left(\frac{E^2}{c^2} - \mathbf{p}^2\right)_{\substack{\text{before} \\ \text{collision}}} &= \left(\frac{E^2}{c^2} - \mathbf{p}^2\right)_{\substack{\text{after} \\ \text{collision}}} \\
\frac{\left(E_{e_1} + E_p\right)^2}{c^2} - \left(\mathbf{p}_{e_1} + \mathbf{p}_p\right)^2 &= \frac{\left(E_{e_2} + E_X\right)^2}{c^2} - \left(\mathbf{p}_{e_2} + \mathbf{p}_X\right)
\end{aligned}$$

Just before the collision, the electron and proton are travelling directly towards each other. Thus, $\left(\mathbf{p}_{e_1} + \mathbf{p}_p\right)^2 = \left(p_{e_1} - p_p\right)^2$. Also, since we wish to find the maximum value attainable for m_X, we set $\mathbf{p}_{e_2} = \mathbf{p}_X = \mathbf{0}$, and $E_{e_2} = m_e c^2$ and $E_X = m_X c^2$, where m_e and m_X are rest masses.

$$\begin{aligned}
\frac{\left(m_e c^2 + m_X c^2\right)^2}{c^2} &= \frac{(E_{e_1} + E_p)^2}{c^2} - (p_{e_1} - p_p)^2 \\
c^4 (m_e + m_X)^2 &= (E_{e_1} + E_p)^2 - \left(\sqrt{E_{e_1}^2 - m_e^2 c^4} - \sqrt{E_p^2 - m_p^2 c^4}\right)^2 \\
&\qquad \text{(using the earlier derived expression for } p) \\
m_X &= \frac{1}{c^2}\sqrt{(E_{e_1} + E_p)^2 - \left(\sqrt{E_{e_1}^2 - m_e^2 c^4} - \sqrt{E_p^2 - m_p^2 c^4}\right)^2} - m_e \\
m_X &\approx 5.671 \cdot 10^{-25} \text{ kg}
\end{aligned}$$

3.

$$\frac{\left(E_{e_1} + E_p\right)^2}{c^2} - \left(\mathbf{p}_{e_1} + \mathbf{p}_p\right)^2 = \frac{\left(E_{e_2} + E_X\right)^2}{c^2} - \left(\mathbf{p}_{e_2} + \mathbf{p}_X\right) \qquad \text{(from before)}$$

Since the proton is at rest before the collision, we have $\left(\mathbf{p}_{e_1} + \mathbf{p}_p\right)^2 = p_{e_1}^2$ and $E_p = m_p c^2$. Since we want to find the minimum energy required for the electron to produce m_X above, we set $\mathbf{p}_{e_2} = \mathbf{p}_X = \mathbf{0}$, and $E_{e_2} = m_e c^2$ and $E_X = m_X c^2$.

$$\begin{aligned}
\frac{\left(E_{e_1} + m_p c^2\right)^2}{c^2} - p_{e_1}^2 &= \frac{\left(m_e c^2 + m_X c^2\right)^2}{c^2} \\
E_{e_1}^2 + m_p^2 c^4 + 2E_{e_1} m_p c^2 - E_{e_1}^2 + m_e^2 c^4 &= c^4 (m_e + m_X)^2 \\
E_{e_1} &= c^2 \frac{(m_e + m_X)^2 - m_p^2 - m_e^2}{2m_p} \\
E_{e_1} &\approx 8.640 \cdot 10^{-6} \text{ J} \\
&\approx 5.393 \cdot 10^4 \text{ GeV}
\end{aligned}$$

4.

$$\begin{aligned}
\Delta E &= \frac{2\pi}{\omega} P \\
&= \frac{2\pi r}{c\beta} \frac{2q^2 c \gamma^4 \beta^4}{3r^2} \\
&= \frac{4\pi q^2 \gamma^4 \beta^3}{3r} \\
&= \frac{4\pi q^2}{3r} \frac{\beta^3}{\left(1 - \beta^2\right)^2}
\end{aligned}$$

$$
\begin{aligned}
p &= \frac{\sqrt{E^2 - m_0^2 c^4}}{c} \\
\implies \gamma m_0 v &= \frac{\sqrt{E^2 - m_0^2 c^4}}{c} \\
\implies \frac{\beta}{\sqrt{1-\beta^2}} &= \frac{\sqrt{E^2 - m_0^2 c^4}}{m_0 c^2} \\
\implies \beta^2 &= \left(1-\beta^2\right)\left(\frac{E^2}{m_0^2 c^4} - 1\right) \\
&= \frac{E^2}{m_0^2 c^4} - 1 - \frac{E^2}{m_0^2 c^4}\beta^2 + \beta^2 \\
\implies \beta^2 &= 1 - \frac{m_0^2 c^4}{E^2}
\end{aligned}
$$

$$
\begin{aligned}
\implies \Delta E &= \frac{4\pi q^2}{3r} \frac{\left(1 - \frac{m_0^2 c^4}{E^2}\right)^{\frac{3}{2}}}{\left(\frac{m_0^2 c^4}{E^2}\right)^2} \\
&= \frac{4\pi q^2}{3r} \left(1 - \frac{m_0^2 c^4}{E^2}\right)^{\frac{3}{2}} \frac{E^4}{m_0^4 c^8}
\end{aligned}
$$

$$
\begin{aligned}
\Delta E_{\text{electron}} &\approx 8.995 \cdot 10^{-22}\ \text{J} \\
&\approx 5.614 \cdot 10^{-12}\ \text{GeV} \\
&\approx 2.042 \cdot 10^{-13}\ E_{\text{electron}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta E_{\text{proton}} &\approx 9.913 \cdot 10^{-29}\ \text{J} \\
&\approx 6.187 \cdot 10^{-19}\ \text{GeV} \\
&\approx 6.725 \cdot 10^{-22}\ E_{\text{proton}}
\end{aligned}
$$

5.

$$
P(t) = \frac{2q^2}{3c}\left\{\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2\right]\right\}_{\text{ret}} \qquad \text{(relativistic Larmor equation)}
$$

$$
P_{\text{linear}}(t) = \frac{2q^2}{3c}\gamma^6\dot{\boldsymbol{\beta}}^2\bigg|_{\text{ret}} \qquad \left(\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}} \implies \boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = \mathbf{0}\right)
$$

6.

Let us write $P_{\text{linear}}(t)$ in terms of $\dot{\mathbf{p}}^2$ using $\mathbf{p} = \gamma m_0 c \boldsymbol{\beta}$.

$$
\begin{aligned}
\mathbf{p} &= \gamma m_0 c \boldsymbol{\beta} \\
\implies \frac{\dot{\mathbf{p}}}{m_0 c} &= \dot{\gamma}\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \left(1-\boldsymbol{\beta}^2\right)^{-\frac{1}{2}} \\
\implies \dot{\gamma} &= -\frac{1}{2}\left(1-\boldsymbol{\beta}^2\right)^{-\frac{3}{2}}\left(-2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right) \\
&= \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\,\gamma^3
\end{aligned}
$$

$$
\begin{aligned}
\implies \frac{\dot{\mathbf{p}}}{m_0 c} &= \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\,\gamma^3\,\boldsymbol{\beta} + \gamma\,\dot{\boldsymbol{\beta}} \\
&= \gamma\left(|\boldsymbol{\beta}||\dot{\boldsymbol{\beta}}|\gamma^2\,\boldsymbol{\beta} + \dot{\boldsymbol{\beta}}\right) && (\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}} \implies \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = |\boldsymbol{\beta}||\dot{\boldsymbol{\beta}}|) \\
\implies \frac{\dot{\mathbf{p}}^2}{\gamma^2 m_0^2 c^2} &= \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}}^2\gamma^4\boldsymbol{\beta}^2 + \dot{\boldsymbol{\beta}}^2 + 2|\boldsymbol{\beta}||\dot{\boldsymbol{\beta}}|\gamma^2\,\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} \\
&= \gamma^4\boldsymbol{\beta}^4\dot{\boldsymbol{\beta}}^2 + \dot{\boldsymbol{\beta}}^2 + 2\gamma^2\boldsymbol{\beta}^2\dot{\boldsymbol{\beta}}^2 && (\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}} \implies \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = |\boldsymbol{\beta}||\dot{\boldsymbol{\beta}}|) \\
&= \left(1 + \gamma^2\boldsymbol{\beta}^2\right)^2\dot{\boldsymbol{\beta}}^2 \\
&= \left(\frac{1 - \boldsymbol{\beta}^2 + \boldsymbol{\beta}^2}{1 - \boldsymbol{\beta}^2}\right)^2\dot{\boldsymbol{\beta}}^2 \\
&= \gamma^4\dot{\boldsymbol{\beta}}^2 \\
\implies \dot{\boldsymbol{\beta}}^2 &= \frac{\dot{\mathbf{p}}^2}{\gamma^6 m_0^2 c^2}
\end{aligned}
$$

$$
\begin{aligned}
\implies P_{\text{linear}}(t) &= \frac{2q^2\gamma^6}{3c}\frac{\dot{\mathbf{p}}^2}{\gamma^6 m_0^2 c^2}\bigg|_{\text{ret}} \\
&= \frac{2q^2}{3m_0^2c^3}\dot{\mathbf{p}}^2\bigg|_{\text{ret}}
\end{aligned}
$$

Now let us consider the circular accelerator.

$$
\begin{aligned}
P(t) &= \frac{2q^2}{3c}\left\{\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2\right]\right\}_{\text{ret}} && \text{(relativistic Larmor equation)} \\
P_{\text{circular}}(t) &= \frac{2q^2}{3c}\left\{\gamma^6\left[\dot{\boldsymbol{\beta}}^2 - \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}}^2\right]\right\}_{\text{ret}} && (\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}} \implies \boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = |\boldsymbol{\beta}||\dot{\boldsymbol{\beta}}|) \\
&= \frac{2q^2}{3c}\gamma^6\dot{\boldsymbol{\beta}}^2\left(1 - \boldsymbol{\beta}^2\right)\bigg|_{\text{ret}} \\
&= \frac{2q^2}{3c}\gamma^4\dot{\boldsymbol{\beta}}^2\bigg|_{\text{ret}}
\end{aligned}
$$

From before we have

$$
\begin{aligned}
\frac{\dot{\mathbf{p}}}{m_0 c} &= \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\,\gamma^3\,\boldsymbol{\beta} + \gamma\,\dot{\boldsymbol{\beta}} \\
&= \gamma\,\dot{\boldsymbol{\beta}} && (\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}} \implies \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \mathbf{0}) \\
\implies \dot{\boldsymbol{\beta}}^2 &= \frac{\dot{\mathbf{p}}^2}{\gamma^2 m_0^2 c^2}
\end{aligned}
$$

$$
\begin{aligned}
\implies P_{\text{circular}}(t) &= \frac{2q^2\gamma^4}{3c}\frac{\dot{\mathbf{p}}^2}{\gamma^2 m_0^2 c^2}\bigg|_{\text{ret}} \\
&= \frac{2q^2}{3m_0^2c^3}\gamma^2\dot{\mathbf{p}}^2\bigg|_{\text{ret}}
\end{aligned}
$$

We thus have $P_{\text{circular}}(t) = \gamma^2(t_{\text{ret}})P_{\text{linear}}(t)$. Since $\gamma^2 > 1$, the radiated power from a circular accelerator is greater than that from a linear accelerator, under the same external force. This is mostly due to the extra power radiated due to the bending of the particle trajectories.

Question 2

1.

In class and for Homework 2 we derived the electric dipole term of the magnetic field in the far-field zone as

$$\mathbf{B}^{\mathrm{ED}}(t,\mathbf{x}) \approx -\frac{1}{rc^2}\,\mathbf{n}\times\ddot{\mathbf{d}}\Big(t-\frac{r}{c}\Big)\,.$$

From the expression $\mathbf{E} = -\mathbf{n}\times\mathbf{B}$, this leads to

$$\mathbf{E}^{\mathrm{ED}}(t,\mathbf{x}) \approx \frac{1}{rc^2}\,\mathbf{n}\times\Big[\mathbf{n}\times\ddot{\mathbf{d}}\Big(t-\frac{r}{c}\Big)\Big]\,.$$

The electric field due to a moving point charge is given by

$$\mathbf{E}_{\mathrm{mpc}}(t,\mathbf{x}) = \frac{q}{r^2}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\mathrm{ret}} + \frac{q}{rc}\left\{\frac{\mathbf{n}\times\Big[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right\}_{\mathrm{ret}}\,.$$

In the far-field zone, the electric field can be approximated by the leading term, i.e. the field due to the electric dipole. We can also approximate the above expression in the far-field zone by the second, more dominant term, i.e. the acceleration field. This leads to

$$\mathbf{E}^{\mathrm{ED}}_{\mathrm{mpc}}(t,\mathbf{x}) \approx \mathbf{E}_{\mathrm{acc}}(t,\mathbf{x}) = \frac{q}{rc}\left\{\frac{\mathbf{n}\times\Big[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right\}_{\mathrm{ret}}\,.$$

To connect $\mathbf{E}^{\mathrm{ED}}$ and $\mathbf{E}^{\mathrm{ED}}_{\mathrm{mpc}}$, we first make the assumption that $\beta = |\boldsymbol{\beta}| \ll 1$, which leads to

$$\begin{aligned}\mathbf{E}_{\mathrm{acc}}(t,\mathbf{x}) &\approx \frac{q}{rc}\Big[\mathbf{n}\times\Big(\mathbf{n}\times\dot{\boldsymbol{\beta}}\Big)\Big]_{\mathrm{ret}} \\ &= \frac{1}{rc^2}\,\mathbf{n}\times\Big[\mathbf{n}\times q\,\ddot{\mathbf{X}}\Big(t-\frac{r}{c}\Big)\Big]\,. \qquad \text{(charge position } \mathbf{X}(t) \neq \text{observer position } \mathbf{x})\end{aligned}$$

Now let us simplify the electric dipole moment $\mathbf{d}$ for a single point charge q.

$$\begin{aligned}\mathbf{d}(t) &\equiv \int_V d^3\mathbf{x}'\,\rho(t,\mathbf{x}')\,\mathbf{x}' \\ &= \int_V d^3\mathbf{x}'\,q\delta(\mathbf{X}(t))\,\mathbf{x}' \\ &= q\,\mathbf{X}(t) \\ \Longrightarrow\ \ddot{\mathbf{d}}(t) &= q\,\ddot{\mathbf{X}}(t)\end{aligned}$$

Thus in the approximation of $\beta \ll 1$, the acceleration field reduces to

$$\mathbf{E}_{\mathrm{acc}}(t,\mathbf{x}) \approx \frac{1}{rc^2}\,\mathbf{n}\times\Big[\mathbf{n}\times\ddot{\mathbf{d}}\Big(t-\frac{r}{c}\Big)\Big] = \mathbf{E}^{\mathrm{ED}}.$$

2.

For simplicity we will let the charge position $\mathbf{X}(t)$ take the form $\mathbf{X}(t) = \rho\,[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}]$, i.e. $t = 0$ corresponds to $\mathbf{X}(0) = (\rho, 0, 0)$.

$$
\begin{aligned}
\mathbf{d}(t) &\equiv \int_V d^3\mathbf{x}'\,\varrho(t,\mathbf{x}')\,\mathbf{x}' && \text{(charge density } \varrho \neq \text{ motion radius } \rho\text{)} \\
&= q\,\mathbf{X}(t) && \text{(point charge)} \\
\mathbf{d}(t) &= q\rho\,[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}] \\
\mathbf{m}(t) &\equiv \frac{1}{2}\int_V d^3\mathbf{x}'\,\mathbf{x}' \times \mathbf{j}(t,\mathbf{x}') \\
&= \frac{1}{2}\,\mathbf{X}(t) \times q\,\dot{\mathbf{X}}(t) && \text{(point charge)} \\
&= \frac{q\rho^2\omega}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \end{vmatrix} \\
&= \frac{q\rho^2\omega}{2}\left[\cos^2(\omega t) + \sin^2(\omega t)\right]\hat{z} \\
\mathbf{m} &= \frac{q\rho^2\omega}{2}\,\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}^{\mathrm{ED}}(t,\mathbf{x}) &= \frac{1}{rc^2}\,\mathbf{n} \times \left[\mathbf{n} \times \ddot{\mathbf{d}}\left(t - \frac{r}{c}\right)\right] \\
&= \frac{q}{rc^2}\,\mathbf{n} \times \left[\mathbf{n} \times \ddot{\mathbf{X}}\left(t - \frac{r}{c}\right)\right] \\
\mathbf{E}^{\mathrm{ED}}(t,\mathbf{x}) &= -\frac{q\rho\omega^2}{rc^2}\,\mathbf{n} \times \{\mathbf{n} \times [\cos(\omega t_{\mathrm{ret}})\,\hat{x} + \sin(\omega t_{\mathrm{ret}})\,\hat{y}]\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}^{\mathrm{MD}}(t,\mathbf{x}) &= \frac{1}{rc^3}\,\mathbf{n} \times \ddot{\mathbf{m}}\left(t - \frac{r}{c}\right) \\
\mathbf{E}^{\mathrm{MD}}(t,\mathbf{x}) &= \mathbf{0}
\end{aligned}
$$

Since the magnetic dipole moment $\ddot{\mathbf{m}}$ is time-independent, the magnetic dipole electric field $\mathbf{E}^{\mathrm{MD}}$ (which depends on the second time derivative of the magnetic dipole moment) vanishes and so is clearly subdominant.

Question 3

1.

$$\begin{aligned}\text{Re}\mathbf{E} &= \mathbf{a_0}\sin(kz-\omega t)\\ &= (a_{0,x}\,\hat{\mathbf{x}} + a_{0,y}\,\hat{\mathbf{y}})\cos\Big(kz-\omega t-\frac{\pi}{2}\Big)\end{aligned}$$

This plane wave is linearly polarised with

$$|E_{0,x}| = a_{0,x} \qquad \varphi = -\frac{\pi}{2}$$
$$|E_{0,y}| = a_{0,y} \qquad \delta = 0$$

$$\begin{aligned}\text{Re}\mathbf{B} &= \frac{1}{\omega}\,\mathbf{k}\times\text{Re}\mathbf{E}\\ &= \frac{k}{\omega}\,\hat{\mathbf{z}}\times\mathbf{a_0}\sin(kz-\omega t)\\ &= \frac{1}{c}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 0 & 1\\ a_{0,x} & a_{0,y} & 0\end{vmatrix}\sin(kz-\omega t)\end{aligned}$$

$$\text{Re}\mathbf{B} = \frac{1}{c}\,\big(-a_{0,y}\,\hat{\mathbf{x}} + a_{0,x}\,\hat{\mathbf{y}}\big)\sin(kz-\omega t)$$

$$\begin{aligned}\text{Re}\mathbf{E} &= a_0\,[\hat{\mathbf{x}}\cos(kz-\omega t) + \hat{\mathbf{y}}\sin(kz-\omega t)]\\ &= a_0\cos(kz-\omega t)\,\hat{\mathbf{x}} + a_0\sin(kz-\omega t)\,\hat{\mathbf{y}}\end{aligned}$$

This plane wave is circularly polarised with

$$|E_{0,x}| = a_0 \qquad \varphi = 0$$
$$|E_{0,y}| = a_0 \qquad \delta = -\frac{\pi}{2}$$

$$\begin{aligned}\text{Re}\mathbf{B} &= \frac{1}{\omega}\,\mathbf{k}\times\text{Re}\mathbf{E}\\ &= \frac{k}{\omega}\,\hat{\mathbf{z}}\times a_0\,[\hat{\mathbf{x}}\cos(kz-\omega t) + \hat{\mathbf{y}}\sin(kz-\omega t)]\\ &= \frac{a_0}{c}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 0 & 1\\ \cos(kz-\omega t) & \sin(kz-\omega t) & 0\end{vmatrix}\end{aligned}$$

$$\text{Re}\mathbf{B} = \frac{a_0}{c}\,[-\sin(kz-\omega t)\,\hat{\mathbf{x}} + \cos(kz-\omega t)\,\hat{\mathbf{y}}]$$

2.

$$\text{Re}\mathbf{E} = \mathbf{a_0}\sin(kz-\omega t)$$
$$\Longrightarrow\ \mathbf{E} = -i\,\mathbf{a_0}\,e^{i(kz-\omega t)}$$
$$\text{Re}\mathbf{B} = \frac{1}{c}\,\big(-a_{0,y}\,\hat{\mathbf{x}} + a_{0,x}\,\hat{\mathbf{y}}\big)\sin(kz-\omega t)$$
$$\Longrightarrow\ \mathbf{B} = \frac{i}{c}\,\big(a_{0,y}\,\hat{\mathbf{x}} - a_{0,x}\,\hat{\mathbf{y}}\big)\,e^{i(kz-\omega t)}$$

$$\text{Re}\mathbf{E} = a_0\,[\cos(kz-\omega t)\,\hat{\mathbf{x}} + \sin(kz-\omega t)\,\hat{\mathbf{y}}]$$
$$\Longrightarrow\ \mathbf{E} = a_0\,(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}})\,e^{i(kz-\omega t)}$$
$$\text{Re}\mathbf{B} = \frac{a_0}{c}\,[-\sin(kz-\omega t)\,\hat{\mathbf{x}} + \cos(kz-\omega t)\,\hat{\mathbf{y}}]$$
$$\Longrightarrow\ \mathbf{B} = \frac{a_0}{c}\,(i\,\hat{\mathbf{x}} + \hat{\mathbf{y}})\,e^{i(kz-\omega t)}$$

3.

$$\mathbf{S} = \frac{1}{\mu}(\mathrm{Re}\mathbf{E}) \times (\mathrm{Re}\mathbf{B})$$
$$= \frac{\sin^2(kz-\omega t)}{\mu c}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_{0,x} & a_{0,y} & 0 \\ -a_{0,y} & a_{0,x} & 0 \end{vmatrix}$$
$$= \frac{\sin^2(kz-\omega t)}{\mu c}\left(a_{0,x}^2 + a_{0,y}^2\right)\hat{\mathbf{z}}$$
$$\mathbf{S} = \frac{\mathbf{a_0}^2 \sin^2(kz-\omega t)}{\mu c}\,\hat{\mathbf{z}}$$

$$\overline{\mathbf{S}} = \frac{1}{T}\int_0^T \mathbf{S}\,dt$$
$$= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\mathbf{a_0}^2 \sin^2(kz-\omega t)}{\mu c}\,\hat{\mathbf{z}}$$
$$= \frac{\omega\,\mathbf{a_0}^2}{2\pi\mu c}\,\hat{\mathbf{z}}\int_0^{\frac{2\pi}{\omega}} \frac{1-\cos(2kz-2\omega t)}{2}\,dt$$
$$= \frac{\omega\,\mathbf{a_0}^2}{4\pi\mu c}\,\hat{\mathbf{z}}\left[t - \frac{\sin(2kz-2\omega t)}{2\omega}\right]_0^{\frac{2\pi}{\omega}}$$
$$= \frac{\omega\,\mathbf{a_0}^2}{4\pi\mu c}\,\hat{\mathbf{z}}\left[\frac{2\pi}{\omega} - \frac{\sin(2kz-4\pi)}{2\omega} + \frac{\sin(2kz)}{2\omega}\right]$$
$$= \frac{\mathbf{a_0}^2}{4\pi\mu c}\,\hat{\mathbf{z}}\left[2\pi - \frac{\sin(2kz)-\sin(2kz)}{2}\right]$$
$$\overline{\mathbf{S}} = \frac{\mathbf{a_0}^2}{2\mu c}\,\hat{\mathbf{z}}$$

$$\mathbf{S} = \frac{1}{\mu}(\mathrm{Re}\mathbf{E}) \times (\mathrm{Re}\mathbf{B})$$
$$= \frac{a_0^2}{\mu c}\left[\cos(kz-\omega t)\,\hat{\mathbf{x}} + \sin(kz-\omega t)\,\hat{\mathbf{y}}\right]$$
$$\times\left[-\sin(kz-\omega t)\,\hat{\mathbf{x}} + \cos(kz-\omega t)\,\hat{\mathbf{y}}\right]$$
$$= \frac{a_0^2}{\mu c}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \cos(kz-\omega t) & \sin(kz-\omega t) & 0 \\ -\sin(kz-\omega t) & \cos(kz-\omega t) & 0 \end{vmatrix}$$
$$= \frac{a_0^2}{\mu c}\left[\cos^2(kz-\omega t) + \sin^2(kz-\omega t)\right]\hat{\mathbf{z}}$$
$$\mathbf{S} = \frac{a_0^2}{\mu c}\,\hat{\mathbf{z}} = \overline{\mathbf{S}}$$

4.

$$\mathbf{F} = \frac{\langle S\rangle A}{c}\,\hat{z} \qquad \text{(normal incidence)}$$
$$= \frac{\langle S\rangle(A\sin\alpha)}{c}\,\hat{z} \qquad \text{(angled incidence)}$$
$$F_\perp = \frac{\langle S\rangle(A\sin\alpha)}{c}\sin\alpha$$
$$= \frac{\langle S\rangle A\sin^2\alpha}{c}$$
$$P = \frac{F_\perp}{A}$$
$$= \frac{\langle S\rangle\sin^2\alpha}{c}$$

$$
\begin{aligned}
S &= |\mathbf{S}| \\
&= \frac{\mathbf{a_0}^2 \sin^2(kz - \omega t)}{\mu c} \\
\implies \langle S \rangle &= \frac{\mathbf{a_0}^2}{2\mu c} \\
P &= \frac{\langle S \rangle \sin^2 \alpha}{c} \\
P &= \frac{\mathbf{a_0}^2 \sin^2 \alpha}{2\mu c^2}
\end{aligned}
$$

$$
\begin{aligned}
S &= |\mathbf{S}| \\
&= \frac{a_0^2}{\mu c} \\
\implies \langle S \rangle &= \frac{a_0^2}{\mu c} \\
P &= \frac{\langle S \rangle \sin^2 \alpha}{c} \\
P &= \frac{a_0^2 \sin^2 \alpha}{\mu c^2}
\end{aligned}
$$