MAU34402: Classical Electrodynamics
Homework 2 due 22/03/2022

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JS Theoretical Physics

Question 1

1.

The first assumption we make is that the maximal linear extent of the volume containing the charge and current distributions is much smaller than the distance from the observer to the volume, i.e. $\left|\vec{x}'\right| \leqslant a \ll r$. This is done in order to simplify the expression for $\left|\vec{R}\right|$ by Taylor expanding about $\left|\vec{x}'\right| = 0$.

The next assumption we make is that $\frac{a}{c}\left|\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right| \ll 1$. This is done in order to simplify the expressions for φ_{ret} and $\vec{A}_{\text{ret}}$ by Taylor expanding the integrands ρ and $\vec{\jmath}$ about $\frac{\vec{n}\cdot\vec{x}'}{c} = 0$.

Next we write $\rho(t,\vec{x})$ as a Fourier transform

$$\rho(t,\vec{x}) = \int e^{i\omega t}\rho_\omega(\vec{x})\,d\omega$$

and assume a maximal frequency $\omega_{\max}$. This then implies that $\frac{a}{c}\left|\omega_{\max}\right| \ll 1$ or, in terms of the corresponding minimum wavelength, $a \ll \lambda_{\min}$.

Finally we assume that the wavelength of the oscillation of the source is much smaller than the distance from the observer to the volume, i.e. $\lambda \ll r$. This is done in order to find a dominant term when calculating the electromagnetic fields.

Starting with the 3-vector we have

$$\begin{aligned}\vec{A}_{\text{ret}}(t,\vec{x}) &= \frac{1}{c}\int d^3\vec{x}'\,\frac{1}{\left|\vec{x}-\vec{x}'\right|}\vec{\jmath}\left(t-\frac{\left|\vec{x}-\vec{x}'\right|}{c},\vec{x}'\right)\\ &= \frac{1}{rc}\int d^3\vec{x}'\left(1+\frac{\vec{x}\cdot\vec{x}'}{r^2}+\dots\right)\left[\vec{\jmath}\left(t-\frac{r}{c},\vec{x}'\right)+\frac{\vec{x}\cdot\vec{x}'}{rc}\dot{\vec{\jmath}}\left(t-\frac{r}{c},\vec{x}'\right)+\dots\right],\end{aligned}$$

where in the second line we expand each term in the integrand about $\left|\vec{x}'\right| = 0$. To obtain $\vec{A}^{\text{ED}}$ we neglect terms of order $\mathcal{O}\left(\frac{a}{r}\right)$ and $\mathcal{O}\left(\frac{a}{\lambda_{\min}}\right)$, i.e. all terms in the above expansions except leading terms. This then leaves us with

$$\vec{A}^{\text{ED}}(t,\vec{x}) = \frac{1}{rc}\int d^3\vec{x}'\,\vec{\jmath}\left(t-\frac{r}{c},\vec{x}'\right).$$

To obtain $\mathbf{B}^{\text{ED}}$ we simply take the curl of the electric dipole 3-vector, i.e.

$$\mathbf{B}^{\text{ED}} = \frac{1}{rc}\nabla\times\int d^3\vec{x}'\,\vec{\jmath}\left(t-\frac{r}{c},\vec{x}'\right)$$

Manipulating this equation using $\dot{\rho}+\nabla\cdot\vec{\jmath} = 0$ we can write the above as

$$\begin{aligned}\mathbf{B}^{\text{ED}} &= \frac{1}{rc}\nabla\times\dot{\vec{d}}\left(t-\frac{r}{c}\right)\\ &= -\frac{1}{r^2c}\vec{n}\times\dot{\vec{d}}\left(t-\frac{r}{c}\right)-\frac{1}{rc^2}\vec{n}\times\ddot{\vec{d}}\left(t-\frac{r}{c}\right)\\ &\approx -\frac{1}{rc^2}\vec{n}\times\ddot{\vec{d}}\left(t-\frac{r}{c}\right)\end{aligned}$$

in the far field zone.

2.

$$
\begin{aligned}
\mathbf{S} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \\
&= \frac{c}{4\pi}\left|\mathbf{E}\right|\left|\mathbf{B}\right|\sin\alpha\,\vec{n} \\
&= \frac{c}{4\pi}\left|\mathbf{B}\right|^2\vec{n} && (\text{since } |\mathbf{E}| = |\mathbf{B}| \text{ and } \mathbf{E}\perp\mathbf{B}) \\
&\approx \frac{c}{4\pi}\left|\mathbf{B}^{\mathrm{ED}}+\mathbf{B}^{\mathrm{MD}}\right|^2\vec{n} \\
&= \frac{c}{4\pi}\left(\left|\mathbf{B}^{\mathrm{ED}}\right|^2+\left|\mathbf{B}^{\mathrm{MD}}\right|^2+2\mathbf{B}^{\mathrm{ED}}\cdot\mathbf{B}^{\mathrm{MD}}\right)\vec{n} \\
&= \frac{c}{4\pi}\left(\left|\mathbf{E}^{\mathrm{ED}}\right|\left|\mathbf{B}^{\mathrm{ED}}\right|\sin\frac{\pi}{2}+\left|\mathbf{E}^{\mathrm{MD}}\right|\left|\mathbf{B}^{\mathrm{MD}}\right|\sin\frac{\pi}{2}-\frac{2}{r^2c^5}\left(\vec{n}\times\ddot{\vec{d}}\right)\cdot\left[\vec{n}\times\left(\vec{n}\times\ddot{\vec{m}}\right)\right]\right)\vec{n} \\
&= \frac{c}{4\pi}\left(\mathbf{E}^{\mathrm{ED}}\times\mathbf{B}^{\mathrm{ED}}+\mathbf{E}^{\mathrm{MD}}\times\mathbf{B}^{\mathrm{MD}}\right)-\frac{1}{2\pi r^2c^4}\ddot{\vec{d}}\cdot\left(\vec{n}\times\ddot{\vec{m}}\right)\vec{n} \\
&= \mathbf{S}^{\mathrm{ED}}+\mathbf{S}^{\mathrm{MD}}+\frac{1}{2\pi r^2c^4}\vec{n}\cdot\left(\ddot{\vec{d}}\times\ddot{\vec{m}}\right)\vec{n} \\
&= \mathbf{S}^{\mathrm{ED}}+\mathbf{S}^{\mathrm{MD}}+\frac{1}{2\pi r^2c^4}\left|\ddot{\vec{d}}\times\ddot{\vec{m}}\right|\cos\theta\,\vec{n}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P} &= \int_{S^2} d^2\vec{r}\cdot\mathbf{S} \\
&\approx \int_{S^2} d^2\vec{r}\cdot\left(\mathbf{S}^{\mathrm{ED}}+\mathbf{S}^{\mathrm{MD}}+\frac{1}{2\pi r^2c^4}\left|\ddot{\vec{d}}\times\ddot{\vec{m}}\right|\cos\theta\,\vec{n}\right) \\
&= \int_{S^2} d^2\vec{r}\cdot\mathbf{S}^{\mathrm{ED}}+\int_{S^2} d^2\vec{r}\cdot\mathbf{S}^{\mathrm{MD}}+\frac{\left|\ddot{\vec{d}}\times\ddot{\vec{m}}\right|}{2\pi r^2c^4}\int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\,d\theta\,d\phi\,\vec{n}\cdot\cos\theta\,\vec{n} \\
&= \mathcal{P}^{\mathrm{ED}}+\mathcal{P}^{\mathrm{MD}}+\frac{\left|\ddot{\vec{d}}\times\ddot{\vec{m}}\right|}{2\pi c^4}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\cos\theta\,d\theta \\
&= \mathcal{P}^{\mathrm{ED}}+\mathcal{P}^{\mathrm{MD}}+\frac{\left|\ddot{\vec{d}}\times\ddot{\vec{m}}\right|}{c^4}\int_0^{0}u\,du && (u=\sin\theta) \\
&= \mathcal{P}^{\mathrm{ED}}+\mathcal{P}^{\mathrm{MD}}
\end{aligned}
$$

Question 2

$$
\begin{aligned}
P(t) &= \frac{2q^2}{3c}\left[\gamma^6\left(\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right)\right]_{\mathrm{ret}} \\
&= \frac{2q^2}{3c}\left[\gamma^6\left(\dot{\vec{\beta}}^2 - \vec{\beta}^2\dot{\vec{\beta}}^2\sin^2\theta\right)\right]_{\mathrm{ret}} && (\vec{\beta},\ \dot{\vec{\beta}} \text{ separated by } \theta) \\
&= \frac{2q^2}{3c}\left[\gamma^6\dot{\vec{\beta}}^2\left(1 - \vec{\beta}^2\sin^2\theta\right)\right]_{\mathrm{ret}} \\
&\approx \frac{2q^2}{3c}\left(\dot{\vec{\beta}}^2\right)_{\mathrm{ret}} && (1 \gg \vec{\beta}^2\sin^2\theta,\ \gamma\approx 1) \\
&= \frac{2q^2}{3c}\left(\frac{\ddot{\vec{x}}^2}{c^2}\right)_{\mathrm{ret}} \\
&= \frac{2}{3c^3}\left(q\ddot{\vec{x}}\right)^2_{\mathrm{ret}} \\
&= \frac{2}{3c^3}\left[\ddot{\vec{d}}\Big(t-\frac{r}{c}\Big)\right]^2
\end{aligned}
$$

using

$$
\begin{aligned}
\vec{d}\Big(t-\frac{r}{c}\Big) &= \int_V d^3\vec{x}'\,\vec{x}'\,\rho\Big(t-\frac{r}{c},\vec{x}'\Big) \\
&= \int_V d^3\vec{x}'\,\vec{x}'\,q\delta\Big(t-\frac{r}{c},\vec{x}\Big) \\
&= q\vec{x}\big|_{\mathrm{ret}}
\end{aligned}
$$

Question 3

The formula for the relativistic Doppler effect is given by

$$
\lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}},
$$

where λ is the wavelength in the source's reference frame, λ' is the wavelength in the observer's reference frame, and $\beta \equiv \frac{v}{c}$. Rearranging this equation for β gives us

$$
\frac{v}{c} = \frac{\lambda'^2 - \lambda^2}{\lambda'^2 + \lambda^2}.
$$

The wavelengths of red and green light are 620-750nm and 495-570nm, respectively.[1] The slowest speed that corresponds to a red light appearing green due to the Doppler effect would occur for the smallest shift in wavelength, i.e. $\lambda = 620$ nm, $\lambda' = 570$ nm. Substituting these values in leads to $\frac{v}{c} = -\frac{595}{7093} \approx -0.084$ or a minimum speed $|v| \approx 90.5 \times 10^6$ km h^{-1}. The maximum speed limit in Ireland is 120 km h^{-1},[2] and so the garda has calculated that I was speeding by travelling at least 750,000 times the speed limit.

[1] T. J. Bruno, P. D. N. Svoronos, "Ultraviolet–Visible Spectrophotometry" in *CRC Handbook of Fundamental Spectroscopic Correlation Charts*, CRC Press, Boca Raton, pp. 9-24, 2005.

[2] RSA, "Speed Limits" in *Rules of the Road*, The O'Brien Press Ltd, Dublin, pp. 111-122, 2020.