MAU34402: Classical Electrodynamics
Homework 1 due 01/03/2022

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JS Theoretical Physics

Question 1

The Lorentz invariants of the system are

$$\mathbf{E}\cdot\mathbf{B} = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 2\,k \end{pmatrix} = 0 \qquad \mathbf{E}^2 - \mathbf{B}^2 = (k^2 + k^2 + 0) - (0 + 0 + 4\,k^2) = -2\,k^2$$

1.

Since $\mathbf{E}' = \mathbf{0}$ can satisfy both $\mathbf{E}'\cdot\mathbf{B}' = 0$ and $\mathbf{E}'^2 - \mathbf{B}'^2 < 0$, we can find a reference frame where the electric field vanishes, provided that the magnetic field does not also vanish in this frame (else $\mathbf{E}'^2 - \mathbf{B}'^2 = 0 \not< 0$). Since there is no reference frame where the magnetic field vanishes (explained in next section), we can definitively say that there is a Lorentz transformation to another reference frame such that the electric field vanishes.

$$\mathbf{E}' = \gamma\,(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\,\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{E})$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}}\left[\begin{pmatrix} k \\ k \\ 0 \end{pmatrix} + \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 2\,k \end{pmatrix}\right] - \frac{1}{\left(1-\beta^2\right)\left(\frac{1}{\sqrt{1-\beta^2}}+1\right)}\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix}\left(\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} \cdot \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}\right)$$

(letting $\mathbf{E}' = \mathbf{0}$, substituting $\mathbf{E}$ and $\mathbf{B}$ as given and $\gamma = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$ with $\beta \equiv |\boldsymbol{\beta}|$)

$$= \frac{k}{\sqrt{1-\beta^2}}\begin{pmatrix} 1+2\,\beta_y \\ 1-2\,\beta_x \\ 0 \end{pmatrix} - \frac{k\,(\beta_x+\beta_y)}{\sqrt{1-\beta^2}+1-\beta^2}\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix}$$

$$\begin{pmatrix} 1+2\,\beta_y \\ 1-2\,\beta_x \\ 0 \end{pmatrix} = \left(\frac{1}{1+\sqrt{1-\beta^2}}\right)(\beta_x+\beta_y)\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix}$$

Taking the z-component of the above expression, we have that $(\beta_x + \beta_y)\,\beta_z = 0$. If $\beta_z \neq 0$ then $\beta_x + \beta_y = 0$, and so we also have that $1 + 2\,\beta_y = 1 - 2\,\beta_x = 0$. This gives us $\beta_x = \frac{1}{2}$, $\beta_y = -\frac{1}{2}$. By inspection, these values for β_x and β_y also satisfy the expressions for $\beta_z = 0$. Thus if we can find $\mathbf{B}'$ in terms of β_z, we can use the invariance condition $\mathbf{E}'^2 - \mathbf{B}'^2 = -2\,k^2 \implies \mathbf{B}'^2 = 2\,k^2$ to solve for β_z.

$$\mathbf{B}' = \gamma\,(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}\,\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{B})$$

$$= \frac{1}{\sqrt{1-\beta^2}}\left[\begin{pmatrix} 0 \\ 0 \\ 2\,k \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \beta_z \end{pmatrix} \times \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}\right] - \frac{1}{\sqrt{1-\beta^2}+1-\beta^2}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \beta_z \end{pmatrix}\left[\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \beta_z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 2\,k \end{pmatrix}\right]$$

$$= \frac{k}{\sqrt{\frac{1}{2}-\beta_z^2}}\begin{pmatrix} \beta_z \\ -\beta_z \\ 1 \end{pmatrix} - \frac{k\,\beta_z}{\sqrt{\frac{1}{2}-\beta_z^2}+\frac{1}{2}-\beta_z^2}\begin{pmatrix} 1 \\ -1 \\ 2\,\beta_z \end{pmatrix}$$

By inspection, $\beta_z = 0$ results in $\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ k\sqrt{2} \end{pmatrix}$, and thus $\mathbf{B}'^2 = 2k^2$ as required. Therefore the transformed field is given by

$$\boldsymbol{\beta} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \qquad \mathbf{E}' = \mathbf{0} \qquad \mathbf{B}' = \begin{pmatrix} 0 \\ 0 \\ k\sqrt{2} \end{pmatrix}$$

2.

Since we have that $\mathbf{E}'^2 - \mathbf{B}'^2 < 0$ in any reference frame, then in order to have a reference frame where $\mathbf{B}' = \mathbf{0}$, we would need to satisfy the requirement $\mathbf{E}'^2 < 0$, which is not possible, and so we can deduce that there is no Lorentz transformation to another reference frame such that the magnetic field vanishes.

3.

Since we have that $\mathbf{E}' \cdot \mathbf{B}' = 0$ in any reference frame, then either the electric field and magnetic field are perpendicular, or one of $\mathbf{E}'$ and $\mathbf{B}'$ are $\mathbf{0}$. Since in both of these cases we have that $\mathbf{E}' \nparallel \mathbf{B}'$, there is no Lorentz transformation to another reference frame such that the electric and magnetic fields are parallel.

Question 2

1.

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$$\begin{aligned} {\Lambda^0}_0 &= \gamma & {\Lambda^0}_k &= -\gamma\,\beta_k = {\Lambda^k}_0 & {\Lambda^k}_k &= 1 + \frac{\gamma - 1}{\beta^2}\,\beta_k^2 & {\Lambda^k}_l &= \frac{\gamma-1}{\beta^2}\,\beta_k\,\beta_l \\ {\Lambda_0}^0 &= g_{0\rho}\,g^{0\sigma}\,{\Lambda^\rho}_\sigma & {\Lambda_0}^k &= g_{0\rho}\,g^{k\sigma}\,{\Lambda^\rho}_\sigma & {\Lambda_k}^k &= g_{k\rho}\,g^{k\sigma}\,{\Lambda^\rho}_\sigma & {\Lambda_k}^l &= g_{k\rho}\,g^{l\sigma}\,{\Lambda^\rho}_\sigma \\ &= g_{00}\,g^{00}\,{\Lambda^0}_0 & &= g_{00}\,g^{kk}\,{\Lambda^0}_k & &= g_{kk}\,g^{kk}\,{\Lambda^k}_k & &= g_{kk}\,g^{ll}\,{\Lambda^k}_l \\ &= \gamma & &= \gamma\,\beta_k & &= 1 + \frac{\gamma-1}{\beta^2}\,\beta_k^2 & &= \frac{\gamma-1}{\beta^2}\,\beta_k\,\beta_l \end{aligned}$$

Since we are considering a Lorentz boost in the y-direction, we have that $\boldsymbol{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}$.

Thus the transformation matrix is given by

$$({\Lambda_\mu}^\nu) = \begin{pmatrix} \gamma & 0 & \gamma\,\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\,\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & 0 & \sinh\zeta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\zeta & 0 & \cosh\zeta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

using

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} & \gamma\,\beta &= \cosh\zeta\,\frac{\sinh\zeta}{\cosh\zeta} \\ &= \frac{1}{\sqrt{1 - \frac{\sinh^2\zeta}{\cosh^2\zeta}}} & &= \sinh\zeta \\ &= \frac{\cosh\zeta}{\sqrt{\cosh^2\zeta - \sinh^2\zeta}} \\ &= \cosh\zeta \end{aligned}$$

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$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad x' = \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1\,\cos\theta + x_3\,\sin\theta \\ x_2 \\ -x_1\,\sin\theta + x_3\,\cos\theta \end{pmatrix}$$

$$x'_\rho = \Lambda_\rho{}^\nu\, x_\nu$$

$$\Lambda_0{}^0 = 1 \qquad \Lambda_1{}^1 = \cos\theta \qquad \Lambda_1{}^3 = \sin\theta \qquad \Lambda_2{}^2 = 1 \qquad \Lambda_3{}^1 = -\sin\theta \qquad \Lambda_3{}^3 = \cos\theta$$

$$\implies (\Lambda_\mu{}^\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

2.

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F'_{\mu\nu} = \Lambda_\mu{}^\rho\,\Lambda_\nu{}^\sigma\,F_{\rho\sigma}$$

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$$\begin{aligned} E'_x &= F'_{01} \\ &= \Lambda_0{}^\rho\,\Lambda_1{}^\sigma\,F_{\rho\sigma} \\ &= \Lambda_0{}^0\,\Lambda_1{}^1\,F_{01} + \Lambda_0{}^2\,\Lambda_1{}^1\,F_{21} + 0 \\ &= \gamma\cdot 1\cdot E_x + \gamma\,\beta\cdot 1\cdot B_z \\ &= \gamma\,(E_x + \beta\,B_z) \end{aligned}$$

$$\begin{aligned} B'_x &= F'_{32} \\ &= \Lambda_3{}^\rho\,\Lambda_2{}^\sigma\,F_{\rho\sigma} \\ &= \Lambda_3{}^3\,\Lambda_2{}^0\,F_{30} + \Lambda_3{}^3\,\Lambda_2{}^2\,F_{32} + 0 \\ &= 1\cdot\gamma\,\beta\cdot(-E_z) + 1\cdot\gamma\cdot B_x \\ &= \gamma\,(B_x - \beta\,E_z) \end{aligned}$$

$$\begin{aligned} E'_y &= F'_{02} \\ &= \Lambda_0{}^\rho\,\Lambda_2{}^\sigma\,F_{\rho\sigma} \\ &= \Lambda_0{}^0\,\Lambda_2{}^2\,F_{02} + \Lambda_0{}^2\,\Lambda_2{}^0\,F_{20} + 0 \\ &= \gamma\cdot\gamma\cdot E_y + \gamma\,\beta\cdot\gamma\,\beta\cdot(-E_y) \\ &= \gamma^2\left(1-\beta^2\right)E_y \\ &= E_y \end{aligned}$$

$$\begin{aligned} B'_y &= F'_{13} \\ &= \Lambda_1{}^\rho\,\Lambda_3{}^\sigma\,F_{13} \\ &= \Lambda_1{}^1\,\Lambda_3{}^3\,F_{13} + 0 \\ &= 1\cdot 1\cdot B_y \\ &= B_y \end{aligned}$$

$$\begin{aligned} E'_z &= F'_{03} \\ &= \Lambda_0{}^\rho\,\Lambda_3{}^\sigma\,F_{\rho\sigma} \\ &= \Lambda_0{}^0\,\Lambda_3{}^3\,F_{03} + \Lambda_0{}^2\,\Lambda_3{}^3\,F_{23} + 0 \\ &= \gamma\cdot 1\cdot E_z + \gamma\,\beta\cdot 1\cdot(-B_x) \\ &= \gamma\,(E_z - \beta\,B_x) \end{aligned}$$

$$\begin{aligned} B'_z &= F'_{21} \\ &= \Lambda_2{}^\rho\,\Lambda_1{}^\sigma\,F_{\rho\sigma} \\ &= \Lambda_2{}^0\,\Lambda_1{}^1\,F_{01} + \Lambda_2{}^2\,\Lambda_1{}^1\,F_{21} + 0 \\ &= \gamma\,\beta\cdot 1\cdot E_x + \gamma\cdot 1\cdot B_z \\ &= \gamma\,(B_z + \beta\,E_x) \end{aligned}$$

$$\mathbf{E}' = \begin{pmatrix} \gamma\,(E_x + \beta\,B_z) \\ E_y \\ \gamma\,(E_z - \beta\,B_x) \end{pmatrix} \qquad \mathbf{B}' = \begin{pmatrix} \gamma\,(B_x - \beta\,E_z) \\ B_y \\ \gamma\,(B_z + \beta\,E_x) \end{pmatrix}$$

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$$\begin{aligned}
E'_x &= F'_{01} \\
&= \Lambda_0{}^\rho \, \Lambda_1{}^\sigma \, F_{\rho\sigma} \\
&= \Lambda_0{}^0 \, \Lambda_1{}^1 \, F_{01} + \Lambda_0{}^0 \, \Lambda_1{}^3 \, F_{03} + 0 \\
&= 1 \cdot \cos\theta \cdot E_x + 1 \cdot \sin\theta \cdot E_z \\
&= E_x \cos\theta + E_z \sin\theta
\end{aligned}$$

$$\begin{aligned}
B'_x &= F'_{32} \\
&= \Lambda_3{}^\rho \, \Lambda_2{}^\sigma \, F_{\rho\sigma} \\
&= \Lambda_3{}^1 \, \Lambda_2{}^2 \, F_{12} + \Lambda_3{}^3 \, \Lambda_2{}^2 \, F_{32} + 0 \\
&= -\sin\theta \cdot 1 \cdot (-B_z) + \cos\theta \cdot 1 \cdot B_x \\
&= B_x \cos\theta + B_z \sin\theta
\end{aligned}$$

$$\begin{aligned}
E'_y &= F'_{02} \\
&= \Lambda_0{}^\rho \, \Lambda_2{}^\sigma \, F_{\rho\sigma} \\
&= \Lambda_0{}^0 \, \Lambda_2{}^2 \, F_{02} + 0 \\
&= 1 \cdot 1 \cdot E_y \\
&= E_y
\end{aligned}$$

$$\begin{aligned}
B'_y &= F'_{13} \\
&= \Lambda_1{}^\rho \, \Lambda_3{}^\sigma \, F_{13} \\
&= \Lambda_1{}^1 \, \Lambda_3{}^3 \, F_{13} + \Lambda_1{}^3 \, \Lambda_3{}^1 \, F_{31} + 0 \\
&= \cos\theta \cdot \cos\theta \cdot B_y + \sin\theta \cdot (-\sin\theta) \cdot (-B_y) \\
&= B_y \left(\cos^2\theta + \sin^2\theta\right) \\
&= B_y
\end{aligned}$$

$$\begin{aligned}
E'_z &= F'_{03} \\
&= \Lambda_0{}^\rho \, \Lambda_3{}^\sigma \, F_{\rho\sigma} \\
&= \Lambda_0{}^0 \, \Lambda_3{}^1 \, F_{01} + \Lambda_0{}^0 \, \Lambda_3{}^3 \, F_{03} + 0 \\
&= 1 \cdot (-\sin\theta) \cdot E_x + 1 \cdot \cos\theta \cdot E_z \\
&= -E_x \sin\theta + E_z \cos\theta
\end{aligned}$$

$$\begin{aligned}
B'_z &= F'_{21} \\
&= \Lambda_2{}^\rho \, \Lambda_1{}^\sigma \, F_{\rho\sigma} \\
&= \Lambda_2{}^2 \, \Lambda_1{}^1 \, F_{21} + \Lambda_2{}^2 \, \Lambda_1{}^3 \, F_{23} + 0 \\
&= 1 \cdot \cos\theta \cdot B_z + 1 \cdot \sin\theta \cdot (-B_x) \\
&= -B_x \sin\theta + B_z \cos\theta
\end{aligned}$$

$$\mathbf{E}' = \begin{pmatrix} E_x \cos\theta + E_z \sin\theta \\ E_y \\ -E_x \sin\theta + E_z \cos\theta \end{pmatrix} \qquad \mathbf{B}' = \begin{pmatrix} B_x \cos\theta + B_z \sin\theta \\ B_y \\ -B_x \sin\theta + B_z \cos\theta \end{pmatrix}$$

Question 3

1.

The electric and magnetic fields generated by a fixed unit charge in a stationary frame are given by

$$\mathbf{E} = \frac{1}{r^2}\,\hat{r} = \frac{\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}}{r^2} \qquad \mathbf{B} = \mathbf{0}$$

using Gaussian units and spherical coordinates for convenience. We also have $\beta = \frac{v}{c} = \frac{1}{5}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{5}{\sqrt{24}}$ for the moving reference frame. We can find the transformed fields using the expressions obtained in Question 2.2.

$$\begin{aligned} E'_x &= \gamma\,(E_x + \beta\,B_z) \\ &= \frac{5}{\sqrt{24}}\left(\frac{\sin\theta\cos\phi}{r^2} + 0\right) \\ &= \frac{5\sin\theta\cos\phi}{r^2\sqrt{24}} \end{aligned} \qquad \begin{aligned} E'_y &= E_y \\ &= \frac{\sin\theta\sin\phi}{r^2} \end{aligned} \qquad \begin{aligned} E'_z &= \gamma\,(E_z - \beta\,B_x) \\ &= \frac{5}{\sqrt{24}}\left(\frac{\cos\theta}{r^2} - 0\right) \\ &= \frac{5\cos\theta}{r^2\sqrt{24}} \end{aligned}$$

$$\begin{aligned} B'_x &= \gamma\,(B_x - \beta\,E_z) \\ &= \frac{5}{\sqrt{24}}\left(0 - \frac{1}{5}\,\frac{\cos\theta}{r^2}\right) \\ &= -\frac{\cos\theta}{r^2\sqrt{24}} \end{aligned} \qquad \begin{aligned} B'_y &= B_y \\ &= 0 \end{aligned} \qquad \begin{aligned} B'_z &= \gamma\,(B_z + \beta\,E_x) \\ &= \frac{5}{\sqrt{24}}\left(0 - \frac{1}{5}\,\frac{\sin\theta\cos\phi}{r^2}\right) \\ &= \frac{\sin\theta\cos\phi}{r^2\sqrt{24}} \end{aligned}$$

$$\mathbf{E}' = \begin{pmatrix} \dfrac{5\sin\theta\cos\phi}{r^2\sqrt{24}} \\[2ex] \dfrac{\sin\theta\sin\phi}{r^2} \\[2ex] \dfrac{5\cos\theta}{r^2\sqrt{24}} \end{pmatrix} = \begin{pmatrix} \dfrac{5\,x}{\sqrt{24}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\[2ex] \dfrac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\[2ex] \dfrac{5\,z}{\sqrt{24}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \end{pmatrix}$$

$$\mathbf{B}' = \begin{pmatrix} -\dfrac{\cos\theta}{r^2\sqrt{24}} \\[2ex] 0 \\[2ex] \dfrac{\sin\theta\cos\phi}{r^2\sqrt{24}} \end{pmatrix} = \begin{pmatrix} -\dfrac{z}{\sqrt{24}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\[2ex] 0 \\[2ex] \dfrac{x}{\sqrt{24}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \end{pmatrix}$$

2.

The Lorentz invariants in the stationary frame are given by

$$\mathbf{E}\cdot\mathbf{B} = \frac{1}{r^2}\cdot 0 = 0 \qquad\qquad \mathbf{E}^2-\mathbf{B}^2 = \frac{1}{r^4} - 0 = \frac{1}{r^4}$$

In the moving frame we have

$$\begin{aligned}\mathbf{E}'\cdot\mathbf{B}' &= \frac{5\sin\theta\cos\phi}{r^2\sqrt{24}}\left(-\frac{\cos\theta}{r^2\sqrt{24}}\right) + \frac{\sin\theta\sin\phi}{r^2}\cdot 0 + \frac{5\cos\theta}{r^2\sqrt{24}}\,\frac{\sin\theta\cos\phi}{r^2\sqrt{24}}\\ &= -\frac{5\sin\theta\cos\theta\cos\phi}{24\,r^4} + 0 + \frac{5\sin\theta\cos\theta\cos\phi}{24\,r^4}\\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{E}^2-\mathbf{B}^2 &= \frac{25\sin^2\theta\cos^2\phi}{24\,r^4} + \frac{\sin^2\theta\sin^2\phi}{r^4} + \frac{25\cos^2\theta}{24\,r^4} - \frac{\cos^2\theta}{24\,r^4} - 0 - \frac{\sin^2\theta\cos^2\phi}{24\,r^4}\\ &= \frac{24\sin^2\theta\cos^2\phi}{24\,r^4} + \frac{\sin^2\theta\sin^2\phi}{r^4} + \frac{24\cos^2\theta}{24\,r^4}\\ &= \frac{1}{r^4}\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right)\\ &= \frac{1}{r^4}\end{aligned}$$

We therefore have

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{E}'\cdot\mathbf{B}' = 0 \qquad\qquad \mathbf{E}^2-\mathbf{B}^2 = \mathbf{E}'^2-\mathbf{B}'^2 = \frac{1}{r^4} = \frac{1}{\left(x^2+y^2+z^2\right)^2}$$

as expected.