PYU33P15: Atomic Physics
Problem Set 2 due 09/12/2021

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Q1

(a)

$$\begin{aligned} 1 &= |\chi| \\ &= |A|\sqrt{3^2+4^2} \\ &= 5\,|A| \\ \implies |A| &= \frac{1}{5} \end{aligned}$$

(b)

$$S_x = \frac{\hbar}{2}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) \qquad S_y = \frac{\hbar}{2}\left(-i\,|1\rangle\langle 2| + i\,|2\rangle\langle 1|\right) \qquad S_z = \frac{\hbar}{2}\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right)$$

$$\begin{aligned} \langle S_x\rangle_\chi &= A^*\left(-3i\,\langle 1| + 4\,\langle 2|\right)\frac{\hbar}{2}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right)A\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= A^*A\,\frac{\hbar}{2}\left(-3i\,\langle 2| + 4\,\langle 1|\right)\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= |A|^2\,\frac{\hbar}{2}\left(-12\,i + 12\,i\right) \\ \langle S_x\rangle_\chi &= 0 \end{aligned}$$

$$\begin{aligned} \langle S_y\rangle_\chi &= A^*\left(-3i\,\langle 1| + 4\,\langle 2|\right)\frac{\hbar}{2}\left(-i\,|1\rangle\langle 2| + i\,|2\rangle\langle 1|\right)A\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= A^*A\,\frac{\hbar}{2}\left(-3\,\langle 2| + 4i\,\langle 1|\right)\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= |A|^2\,\frac{\hbar}{2}\left(-12 - 12\right) \\ &= \frac{1}{25}\,\frac{\hbar}{2}\left(-24\right) \\ \langle S_y\rangle_\chi &= -\frac{12\,\hbar}{25} \end{aligned}$$

$$\begin{aligned} \langle S_z\rangle_\chi &= A^*\left(-3i\,\langle 1| + 4\,\langle 2|\right)\frac{\hbar}{2}\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right)A\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= A^*A\,\frac{\hbar}{2}\left(-3i\,\langle 1| - 4\,\langle 2|\right)\left(3i\,|1\rangle + 4\,|2\rangle\right) \\ &= |A|^2\,\frac{\hbar}{2}\left(9 - 16\right) \\ &= \frac{1}{25}\,\frac{\hbar}{2}\left(-7\right) \\ \langle S_z\rangle_\chi &= -\frac{7\hbar}{50} \end{aligned}$$

(c)

$$\begin{aligned} S_i^2 &= \left(\frac{\hbar}{2}\,\sigma_i\right)^2 \\ &= \frac{\hbar^2}{4}\,I \\ \implies \left\langle S_i^2 \right\rangle_\chi &= \chi^*\,\frac{\hbar^2}{4}\,I\,\chi \\ &= \frac{\hbar^2}{4}\,|\chi|^2 \\ &= \frac{\hbar^2}{4} \\ \implies \left\langle S_x^2 \right\rangle_\chi = \left\langle S_y^2 \right\rangle_\chi = \left\langle S_z^2 \right\rangle_\chi &= \frac{\hbar^2}{4} \end{aligned}$$

$$\begin{aligned} \Delta S_x &= \sqrt{\langle S_x^2\rangle_\chi - \langle S_x\rangle_\chi^2} \\ &= \sqrt{\frac{\hbar^2}{4} - 0^2} \\ \Delta S_x &= \frac{\hbar}{2} \end{aligned}$$

$$\begin{aligned} \Delta S_y &= \sqrt{\langle S_y^2\rangle_\chi - \langle S_y\rangle_\chi^2} \\ &= \sqrt{\frac{\hbar^2}{4} - \frac{144\,\hbar^2}{625}} \\ \Delta S_y &= \frac{7\,\hbar}{50} \end{aligned}$$

$$\begin{aligned} \Delta S_z &= \sqrt{\langle S_z^2\rangle_\chi - \langle S_z\rangle_\chi^2} \\ &= \sqrt{\frac{\hbar^2}{4} - \frac{49\,\hbar^2}{2500}} \\ \Delta S_z &= \frac{12\,\hbar}{25} \end{aligned}$$

(d)

$$\text{Uncertainty relation: } \Delta\hat{A}\,\Delta\hat{B} \geqslant \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|$$

$$[S_i, S_i] = 0 \implies \text{relation holds for } S_i \text{ and } S_i$$

$$\begin{aligned} \Delta S_x\,\Delta S_y &= \frac{\hbar}{2}\,\frac{7\,\hbar}{50} \\ &= \frac{7\,\hbar^2}{100} \\ \frac{1}{2}\,|\langle[S_x,S_y]\rangle| &= \frac{1}{2}\,|\langle i\,\hbar\,S_z\rangle| \\ &= \frac{1}{2}\left|i\,\hbar\left(-\frac{7\,\hbar}{50}\right)\right| \\ &= \frac{7\,\hbar^2}{100} \\ \Delta S_x\,\Delta S_y &= \frac{1}{2}\,|\langle[S_x,S_y]\rangle| \end{aligned}$$

$$\begin{aligned} \Delta S_y\,\Delta S_z &= \frac{7\,\hbar}{50}\,\frac{12\,\hbar}{25} \\ &= \frac{42\,\hbar^2}{625} \\ \frac{1}{2}\,|\langle[S_y,S_z]\rangle| &= \frac{1}{2}\,|\langle i\,\hbar\,S_x\rangle| \\ &= \frac{1}{2}\,|i\,\hbar\,(0)| \\ &= 0 \\ \Delta S_y\,\Delta S_z &> \frac{1}{2}\,|\langle[S_y,S_z]\rangle| \end{aligned}$$

$$\begin{aligned} \Delta S_z\,\Delta S_x &= \frac{12\,\hbar}{25}\,\frac{\hbar}{2} \\ &= \frac{6\,\hbar}{25} \\ \frac{1}{2}\,|\langle[S_z,S_x]\rangle| &= \frac{1}{2}\,|\langle i\,\hbar\,S_y\rangle| \\ &= \frac{1}{2}\left|i\,\hbar\left(-\frac{12\,\hbar}{25}\right)\right| \\ &= \frac{6\,\hbar^2}{25} \\ \Delta S_z\,\Delta S_x &= \frac{1}{2}\,|\langle[S_z,S_x]\rangle| \end{aligned}$$

Thus the results in (c) are consistent with uncertainty relations.

Q2

(a)

$$\begin{aligned} 0 &= \det(S_y - \lambda\, I) \\ &= \begin{vmatrix} -\lambda & -\frac{i\,\hbar}{2} \\ \frac{i\,\hbar}{2} & -\lambda \end{vmatrix} \\ &= \lambda^2 - \frac{\hbar^2}{4} \\ \implies \lambda &= \pm\frac{\hbar}{2} \end{aligned}$$

$$\begin{aligned} S_y\,\vec{x} &= \lambda\,\vec{x} \\ \implies 0 &= (S_y - \lambda\, I)\,\vec{x} \\ &= \begin{pmatrix} \mp\frac{\hbar}{2} & -\frac{i\,\hbar}{2} \\ \frac{i\,\hbar}{2} & \mp\frac{\hbar}{2} \end{pmatrix}\vec{x} \\ &= -\frac{\hbar}{2}\begin{pmatrix} \pm 1 & i \\ -i & \pm 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \implies 0 &= \pm x_1 + i\,x_2 \\ \implies x_2 &= \pm i\,x_1 \\ \implies \vec{x} &= A\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \\ \vec{x}\text{ normalised} \implies \vec{x} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \end{aligned}$$

Thus the eigenvectors and corresponding eigenvalues of S_y are

$$\vec{x}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \vec{x}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\lambda_+ = \frac{\hbar}{2} \qquad \lambda_- = -\frac{\hbar}{2}$$

(b)

$$
\begin{aligned}
\vec{\chi} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \gamma\,\vec{x}_+ + \delta\,\vec{x}_- \qquad \text{(expressing } \chi \text{ as a linear combination of the eigenvectors of } S_y\text{)} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma + \delta \\ i\,(\gamma - \delta) \end{pmatrix} \\
\implies \alpha &= \frac{\gamma + \delta}{\sqrt{2}},\ \beta = \frac{i\,(\gamma - \delta)}{\sqrt{2}} \\
\implies \gamma &= \frac{\alpha - i\,\beta}{\sqrt{2}},\ \delta = \frac{\alpha + i\,\beta}{\sqrt{2}} \\
\implies \vec{\chi} &= \frac{\alpha - i\,\beta}{\sqrt{2}}\,\vec{x}_+ + \frac{\alpha + i\,\beta}{\sqrt{2}}\,\vec{x}_-
\end{aligned}
$$

$$
\begin{aligned}
P\left(\frac{\hbar}{2}\right) &= \langle \chi \,|\, x_+ \rangle \, \langle x_+ \,|\, \chi \rangle & P\left(-\frac{\hbar}{2}\right) &= \langle \chi \,|\, x_- \rangle \, \langle x_- \,|\, \chi \rangle \\
&= \left(\frac{\alpha - i\,\beta}{\sqrt{2}}\right)^* \frac{\alpha - i\,\beta}{\sqrt{2}} + 0 & &= 0 + \left(\frac{\alpha + i\,\beta}{\sqrt{2}}\right)^* \frac{\alpha + i\,\beta}{\sqrt{2}} \quad \text{(since } \vec{x}_+,\, \vec{x}_- \text{ are orthogonal)} \\
&= \frac{|\alpha - i\,\beta|^2}{2} & &= \frac{|\alpha + i\,\beta|^2}{2}
\end{aligned}
$$

Thus for a state χ given by $\vec{\chi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, the probability of measuring $\frac{\hbar}{2}$ is $\frac{|\alpha - i\,\beta|^2}{2}$, and the probability of measuring $-\frac{\hbar}{2}$ is $\frac{|\alpha + i\,\beta|^2}{2}$.

The sum of these probabilities is $\frac{|\alpha - i\,\beta|^2}{2} + \frac{|\alpha + i\,\beta|^2}{2} = |\gamma|^2 + |\delta|^2$. Since $\vec{\chi}$ is normalised, we then have that $|\vec{\chi}|^2 = |\gamma|^2 + |\delta|^2 = 1$, since $\vec{x}_+$ and $\vec{x}_-$ are orthonormal vectors. Therefore the probabilities $P\left(\frac{\hbar}{2}\right)$ and $P\left(-\frac{\hbar}{2}\right)$ add to 1.

(c)

We have from before that $S_y^2 = \frac{\hbar^2}{4}\, I$, and thus its only eigenvalue is $\frac{\hbar^2}{4}$, corresponding to any non-zero vector as its eigenvector. Thus, for a given state χ, $\vec{\chi}$ will be an eigenvector of S_y^2. Therefore,

$$
\begin{aligned}
P\left(\frac{\hbar^2}{4}\right) &= \langle \chi \,|\, \chi \rangle \, \langle \chi \,|\, \chi \rangle \\
&= 1 \text{ if } \vec{\chi} \text{ is normalised,}
\end{aligned}
$$

and so the probability of measuring $\frac{\hbar^2}{4}$ is 1.

Q3

$$
\begin{aligned}
\vec{\mu}_L &= -\frac{e}{2\,m_e}\,\vec{L} \\
|\mu_L| &= \left|-\frac{e}{2\,m_e}\,\vec{L}\right| \\
&= \frac{e}{2\,m_e}\sqrt{\hbar^2\,\ell\,(\ell-1)} \\
&= \frac{\hbar\,e}{2\,m_e}\,\sqrt{\ell\,(\ell-1)} \\
p\text{-orbital} \implies \ell &= 2 \\
\implies |\mu_L| &= \frac{\hbar\,e}{2\,m_e}\,\sqrt{2\,(2-1)} \\
&= \frac{\hbar\,e}{m_e\sqrt{2}} \approx 1.311\times10^{-23}\ \mathrm{A\cdot m^2}
\end{aligned}
$$

Q4

(a)

$$
\begin{aligned}
E_n &= \frac{E_1}{n^2} \\
\Delta E_{n\to m} &= E_n - E_m \\
&= E_1\left(\frac{1}{n^2}-\frac{1}{m^2}\right) \\
\Delta E_{3\to2} &= E_1\left(\frac{1}{3^2}-\frac{1}{2^2}\right) \\
&= -\frac{5\,E_1}{36} \\
\lambda_{3\to2} &= \frac{h\,c}{\Delta E_{3\to2}} \\
\lambda_{3\to2} &= -\frac{36\,h\,c}{5\,E_1} \approx 656.470\ \mathrm{nm} \\
f_{3\to2} &= \frac{c}{\lambda_{3\to2}} \\
f_{3\to2} &= -\frac{5\,E_1}{36\,h} \approx 4.567\times10^{14}\ \mathrm{Hz}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E'_{\mathrm{r}} + E'_{\mathrm{SO}} &= \frac{E_n^2}{2\,m\,c^2}\left(3 - \frac{4\,n}{l+\frac{1}{2}} + \frac{2\,n\left(j\,(j+1) - l\,(l+1) - \frac{3}{4}\right)}{l\left(l+\frac{1}{2}\right)(l+1)}\right) \\
&= \frac{E_n^2}{2\,m\,c^2}\left(3 + \frac{-4\,n\,l\,(l+1) + 2\,n\,j\,(j+1) - 2\,n\,l\,(l+1) - \frac{3}{2}\,n}{l\left(l+\frac{1}{2}\right)(l+1)}\right) \\
&= \frac{E_n^2}{2\,m\,c^2}\left(3 + \frac{2\,n\,j\,(j+1) - 6\,n\,l\,(l+1) - \frac{3}{2}\,n}{l\left(l+\frac{1}{2}\right)(l+1)}\right) \\
&= \frac{E_n^2}{2\,m\,c^2}\left(3 + \frac{4\,n}{j+\frac{1}{2}}\,\frac{\frac{1}{2}\,j\,(j+1) - \frac{3}{2}\,l\,(l+1) - \frac{3}{8}}{l\left(l+\frac{1}{2}\right)(l+1)}\left(j+\frac{1}{2}\right)\right) \\
&= \frac{E_n^2}{2\,m\,c^2}\left(3 + \frac{4\,n}{j+\frac{1}{2}}\,f(j,l)\right)
\end{aligned}
$$

$$
j = l + s = l \pm \frac{1}{2}
$$

$$
\begin{aligned}
f\left(l+\frac{1}{2}, l\right) &= \frac{\frac{1}{2}\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) - \frac{3}{2}\,l\,(l+1) - \frac{3}{8}}{l\left(l+\frac{1}{2}\right)(l+1)}\,(l+1) \\
&= \frac{\frac{1}{2}\left(l^2 + 2\,l + \frac{3}{4}\right) - \frac{3}{2}\left(l^2 + l\right) - \frac{3}{8}}{l\left(l+\frac{1}{2}\right)} \\
&= \frac{\frac{1}{2}\,l^2 + l + \frac{3}{8} - \frac{3}{2}\,l^2 - \frac{3}{2}\,l - \frac{3}{8}}{l^2 + \frac{1}{2}\,l} \\
&= \frac{-l^2 - \frac{1}{2}\,l}{l^2 + \frac{1}{2}\,l} \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
f\left(l-\frac{1}{2}, l\right) &= \frac{\frac{1}{2}\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right) - \frac{3}{2}\,l\,(l+1) - \frac{3}{8}}{l\left(l+\frac{1}{2}\right)(l+1)}\,l \\
&= \frac{\frac{1}{2}\left(l^2 - \frac{1}{4}\right) - \frac{3}{2}\left(l^2 + l\right) - \frac{3}{8}}{\left(l+\frac{1}{2}\right)(l+1)} \\
&= \frac{\frac{1}{2}\,l^2 - \frac{1}{8} - \frac{3}{2}\,l^2 - \frac{3}{2}\,l - \frac{3}{8}}{l^2 + \frac{3}{2}\,l + \frac{1}{2}} \\
&= \frac{-l^2 - \frac{3}{2}\,l - \frac{1}{2}}{l^2 + \frac{3}{2}\,l + \frac{1}{2}} \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
&\implies f(j,l) = -1 \text{ for } j = l \pm \frac{1}{2} \\
&\implies E'_{\mathrm{r}} + E'_{\mathrm{SO}} = \frac{E_n^2}{2\,m\,c^2}\left(3 - \frac{4\,n}{j+\frac{1}{2}}\right)
\end{aligned}
$$

(c)

(i)

For $n = 2$, we have that $j = l + s = 0 \pm \frac{1}{2}, 1 \pm \frac{1}{2} = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. For $n = 3$, we have that $j = l + s = 0 \pm \frac{1}{2}, 1 \pm \frac{1}{2}, 2 \pm \frac{1}{2} = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. However, from inspecting the correction term, a singularity arises whenever $j = -\frac{1}{2}$, and thus the allowable values of j are $\frac{1}{2}$ and $\frac{3}{2}$ for $n = 2$, and $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$ for $n = 3$. Therefore there are six possible lines in the Balmer series, for the energy transition $3 \to 2$.

(ii)

$$
\begin{aligned}
E_n &= \frac{E_1}{n^2} + \frac{E_1^2}{2\,n^4\,m\,c^2}\left(3 - \frac{4\,n}{j_n + \frac{1}{2}}\right) \\
\Delta E_{j_2,j_3} &= E_3 - E_2 \\
&= \frac{E_1}{3^2} + \frac{E_1^2}{2\,(3^4)\,m\,c^2}\left(3 - \frac{4\,(3)}{j_3 + \frac{1}{2}}\right) - \frac{E_1}{2^2} - \frac{E_1^2}{2\,(2^4)\,m\,c^2}\left(3 - \frac{4\,(2)}{j_2 + \frac{1}{2}}\right) \\
&= -\frac{5\,E_1}{36} + \frac{E_1^2}{m\,c^2}\left(\frac{1}{4\left(j_2 + \frac{1}{2}\right)} - \frac{2}{27\left(j_3 + \frac{1}{2}\right)} - \frac{65}{864}\right) \\
&= -\frac{5\,E_1}{36} + \frac{E_1^2}{m\,c^2}\,g(j_2, j_3)
\end{aligned}
$$

$$g\left(\frac{1}{2},\frac{1}{2}\right) = \frac{29}{288} \qquad g\left(\frac{1}{2},\frac{3}{2}\right) = \frac{119}{864} \qquad g\left(\frac{1}{2},\frac{5}{2}\right) = \frac{389}{2592}$$

$$g\left(\frac{3}{2},\frac{1}{2}\right) = -\frac{7}{288} \qquad g\left(\frac{3}{2},\frac{3}{2}\right) = \frac{11}{864} \qquad g\left(\frac{3}{2},\frac{5}{2}\right) = \frac{65}{2592}$$

$$\lambda_{j_2,j_3} = \frac{h\,c}{\Delta E_{j_2,j_3}}$$

$$\lambda_{\frac{1}{2},\frac{1}{2}} = 656.45693\text{ nm} \qquad \lambda_{\frac{1}{2},\frac{3}{2}} = 656.45227\text{ nm} \qquad \lambda_{\frac{1}{2},\frac{5}{2}} = 656.45072\text{ nm}$$

$$\lambda_{\frac{3}{2},\frac{1}{2}} = 656.47266\text{ nm} \qquad \lambda_{\frac{3}{2},\frac{3}{2}} = 656.46800\text{ nm} \qquad \lambda_{\frac{3}{2},\frac{5}{2}} = 656.46645\text{ nm}$$

$$f_{j_2,j_3} = \frac{\Delta E_{j_2,j_3}}{h}$$

$$f_{\frac{1}{2},\frac{1}{2}} = 4.5668260\times10^{14}\text{ Hz} \qquad f_{\frac{1}{2},\frac{3}{2}} = 4.5668584\times10^{14}\text{ Hz} \qquad f_{\frac{1}{2},\frac{5}{2}} = 4.5668692\times10^{14}\text{ Hz}$$

$$f_{\frac{3}{2},\frac{1}{2}} = 4.5667166\times10^{14}\text{ Hz} \qquad f_{\frac{3}{2},\frac{3}{2}} = 4.5667490\times10^{14}\text{ Hz} \qquad f_{\frac{3}{2},\frac{5}{2}} = 4.5667598\times10^{14}\text{ Hz}$$

$j_2 = \frac{1}{2}, j_3 = \frac{5}{2}$		$j_2 = \frac{1}{2}, j_3 = \frac{3}{2}$		$j_2 = \frac{1}{2}, j_3 = \frac{1}{2}$		$j_2 = \frac{3}{2}, j_3 = \frac{5}{2}$		$j_2 = \frac{3}{2}, j_3 = \frac{3}{2}$		$j_2 = \frac{3}{2}, j_3 = \frac{1}{2}$
	$\Delta\lambda = 0.00155$ nm $\Delta f = 1.08\times10^9$ Hz		$\Delta\lambda = 0.00466$ nm $\Delta f = 3.24\times10^9$ Hz		$\Delta\lambda = 0.00952$ nm $\Delta f = 6.62\times10^9$ Hz		$\Delta\lambda = 0.00155$ nm $\Delta f = 1.08\times10^9$ Hz		$\Delta\lambda = 0.00466$ nm $\Delta f = 3.24\times10^9$ Hz	

Q5

$$
\begin{aligned}
\langle H \rangle &= \left(2\,Z^2 - 4\,Z\,(Z - Z_0) - \frac{5}{4}\,Z\right) E_1 \\
&= \left(-2\,Z^2 + \left(4\,Z_0 - \frac{5}{4}\right) Z\right) E_1
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{d\,\langle H \rangle}{dZ} \\
&= \left(-4\,Z + 4\,Z_0 - \frac{5}{4}\right) E_1 \\
\implies Z &= Z_0 - \frac{5}{16}
\end{aligned}
$$

$$
\begin{aligned}
\implies \langle H \rangle &= \left(-2\left(Z_0 - \frac{5}{16}\right)^2 + \left(4Z_0 - \frac{5}{4}\right)\left(Z_0 - \frac{5}{16}\right)\right) E_1 \\
&= \left(-2\left(Z_0 - \frac{5}{16}\right)^2 + 4\left(Z_0 - \frac{5}{16}\right)^2\right) E_1 \\
&= 2\,E_1 \left(Z_0 - \frac{5}{16}\right)^2 \\
&\approx -196.431\text{ eV for } Z_0 = 3
\end{aligned}
$$

The experimental ground state of energy is -198.09 eV,[1] and so the calculated value deviates less than 1% from the experimental value. This is much lower than the deviation that would be calculated assuming the electrons have no interactions, indicating that the variational approach is a good method of solving ground state energies.

[1]Kramida, A., Ralchenko, Yu., Reader, J., and NIST ASD Team (2021). NIST Atomic Spectra Database (ver. 5.9), [Online]. Available: https://physics.nist.gov/asd [2021, December 6]. National Institute of Standards and Technology, Gaithersburg, MD. DOI: https://doi.org/10.18434/T4W30F